ON CHARACTERS OF CHEVALLEY GROUPS VANISHING AT THE NON-SEMISIMPLE ELEMENTS

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Dedicated to Lino Di Martino on the occasion of his 65th birthday

ABSTRACT. Let G be a finite simple group of Lie type. In this paper we study characters of G that vanish at the non-semisimple elements and whose degree is equal to the order of a maximal unipotent subgroup of G. Such characters can be viewed as a natural generalization of the Steinberg character. For groups G of small rank we also determine the characters of this degree vanishing only at the non-identity unipotent elements.

1. INTRODUCTION

Let G be a finite simple group of Lie type in defining characteristic p, and let $|G|_p$ denote the order of a Sylow p-subgroup of G. There is a unique irreducible representation of G of dimension $|G|_p$ over the complex numbers, and this is called *the Steinberg representation*. We denote here by St the character of the Steinberg representation. The Steinberg character plays a prominent role in the character theory of groups of Lie type, so this encourages one to look for generalizations of such character. In particular, in this paper we consider p-vanishing characters of degree $|G|_p$.

Let H be any finite group and p a prime. Element $x \in H$ of order divisible by p are called p-singular. Characters (in general reducible) that vanish at the p-singular elements of H are called *p-vanishing characters* in this paper. The characters of projective modules over a field of characteristic p are p-vanishing. We think that p-vanishing characters are of significant interest due to the connection with projective modules. In the case where H is p-solvable, the well known result of Fong (1962) tells us that every p-vanishing character is the character of a projective module [6, 32.17]. In general, this is not true. To the best of our knowledge, no publications on p-vanishing characters appears since 1962.

As St is the character of a projective module, one may think of the characters of projective modules of dimension $|G|_p$ as natural analogs of the Steinberg character. Malle and Weigel [20] determined the projective modules of degree $|G|_p$ whose character has the trivial character 1_G as a constituent. If G is a group of Lie type in characteristic p, the second named author showed in [28] that the restriction on 1_G can be dropped. In other words, the Steinberg character and the characters obtained in [20] are the only characters of projective modules of G of degree $|G|_p$. (Note that for other primes dividing $|G|$ and for other groups the problem remains open.) In both the papers [20] and [28] the argument relies on the p-modular

representation theory. Our results on p-vanishing characters of degree $|G|_p$ are stronger, and make no use of the theory of modular representations. Our main result is the following theorem:

Theorem 1.1. *Let G be a simple group of Lie type in defining characteristic p. If $G \in \{A_n(q),\ n > 4,\ {}^2A_n(q),\ n > 2,\ B_n(q),\ q\ odd,\ n > 5,\ C_n(q),\ n > 2,\ D_n^+(q),\ n > 5,\ D_n^-(q),\ n > 3,\ {}^3D_4(q),\ E_6(q),\ {}^2E_6(q),\ E_7(q),\ E_8(q),\ F_4(q),\ {}^2F_4(q^2),\ G_2(q)\}$ then the Steinberg character is the only p-vanishing character of degree $|G|_p$ (in particular every projective module of this dimension is the Steinberg module).*

Naturally, one wishes to know whether the converse is true. Unfortunately, we have not been able to manage with the groups $B_3(q)$, $B_4(q)$, $B_5(q)$, $D_4(q)$ and $D_5(q)$ for arbitrary q. For all other groups we have the full answer:

Theorem 1.2. *Let G be a simple group of Lie type in defining characteristic p.*

(1) *Groups $A_1(q)$, $A_2(q)$, ${}^2B_2(q^2)$, ${}^2G_2(q^2)$ have reducible p-vanishing characters of degree $|G|_p$, where q is any p-power in the former two cases, whereas in the latter two cases q^2 is an odd p-power for $p = 2, 3$, respectively).*

(2) *Groups ${}^2A_2(q)$, $A_3(q)$, $A_4(q)$ have reducible p-vanishing characters of degree $|G|_p$ if and only if $q + 1$ is divisible by 3.*

(3) *Groups $C_2(q) \cong B_2(q)$ have reducible p-vanishing characters of degree $|G|_p$ if and only if $q + 1$ is divisible by 7.*

In fact, we give an explicit decomposition of every p-vanishing character as a sum of irreducible ones, and this is used in the proof of Theorem 1.1. For instance, in order to prove that $PSL(6, q)$ belongs to the list of Theorem 1.1, we need to know all irreducible constituents of the p-vanishing characters of degree q^{10} for the group $PSL(5, q)$.

As long as we have obtained the decompositions of p-vanishing characters in items (1), (2), (3) of Theorem 1.2 in terms of irreducible ones, it is not hard to compute the values of every p-vanishing character and compare it with those of the Steinberg character. (Full information is tabulated at the end of the paper.) We find out that every such character has real values and does not vanish at the semisimple elements. An essential feature of our results is that they are uniform with respect of q.

One of possible applications of our results to projective modules and the structure of the decomposition matrices can be outlined as follows. Let Ψ, Φ be the characters of projective modules; we write $\Psi > \Phi$ if $\Psi - \Phi$ is a proper character. Looking at known decomposition matrices one observes a lot of examples of projective indecomposable modules whose characters Ψ, Φ satisfy $\Psi > \Phi$. However, their dimensions cannot differ by $|G|_p$ for most groups:

Corollary 1.3. *Let G be a group as in* Theorem 1.1. *Let $\Psi > \Phi$ be the characters of two projective modules of G. Then either $\Psi - \Phi = St$ or $\Psi(1) - \Phi(1) \geq 2|G|_p$.*

If Ψ, Φ are the characters of indecomposable projective modules, then $\Psi > \Phi$ means that, for every row of the decomposition matrix, the Φ-entry is not greater than the Ψ-entry. The above corollary implies $\Psi(1) - \Phi(1) \geq 2|G|_p$ (as the first option of the corollary cannot occur).

There are publications concerning generalization of the notion of the Steinberg character in directions other than the one exploited in this paper. Recall that

$St(g) = 0$ if g is not semisimple, otherwise $St(g) = \pm |C_G(g)|_p$. In [11] Walter Feit stated the following question (1993):

Let G be a finite simple group and p a prime divisor of $|G|$. Does there exist any irreducible character of G vanishing at the p-singular elements, other than the Steinberg character for G of Lie type in defining characteristic p?

He conjectured that the Steinberg character is the only one with the above property. The conjecture was later proved to be true by Tiep [26] (by Darafsheh [8] in a special case). Further progress along this line was marked by completing the classification of prime power degree irreducible representations of quasisimple groups [26, 21, 1].

Our proofs of Theorems 1.1 and 1.2 is based on the contemporary theory of characters of groups of Lie type. Particularly important role is played by the Gelfand-Graev characters and Harish-Chandra theory, together with regular and cuspidal characters. One of the main tools in our analysis is the Harish-Chandra restriction/induction technique. One observes that if χ is a p-vanishing character then so is the Harish-Chandra restriction $\overline{\chi}_L$ of χ to every Levi subgroup L of G (see [20, 28] or Lemma 2.5 below). Moreover, $\chi(1)/|G|_p = \overline{\chi}_L(1)/|L|_p$. Let L_u be the subgroup of L generated by the unipotent elements of L. Then, the restriction $\tilde{\chi}$ of $\overline{\chi}_L$ to L_u is a p-vanishing character of L_u of the same degree. Therefore, through induction assumption we get information on properties of $\tilde{\chi}$ for all Levi subgroups L of G.

Another tool used heavily in the proof of Theorem 1.2 is induction on the BN-pair rank of G. So we first prove the theorem for groups of BN-pair rank 1, see Section 5. Surprisingly, only the case with $G = SL(2, q)$ is straightforward. The other groups $^2B_2(q^2)$, $^2G_2(q^2)$ and $SU(3, q)$ require considerable computational work. As the character tables of these groups are available, we would be able to perform required computations manually. However, it is much more efficient to attract the computer program package CHEVIE, created by a research group at University of Aachen.

One could expect that these groups constitute the base of induction for our proof. However, this is not the case as the situation is more complex. Most groups of rank 2, especially, $Sp(4, q)$, $^3D_4(q)$ and $^2F_4(q^2)$ require quite a lot of computational work. Probably, performing this work manually is not realistic, but with use of CHEVIE we complete the analysis of groups of rank 2 (see Section 7).

Our results on groups of rank 2 are sufficient to run induction for unitary groups and $D_n^-(q)$. However, for the series $SL(n, q)$ we cannot run induction from $n < 6$, and for the series $Sp(2n, q)$, the induction starts from $n = 3$. Groups $B_3(q)$ and $D_4^+(q)$ should also be handled by computations. However, the CHEVIE package does not contain enough data to deal with these groups. We are still unable to complete the work for groups $B_n(q)$, $n = 3, 4, 5$, and $D_n^+(q)$, $n = 4, 5$. If $n > 5$, as well as for the groups $E_6(q), E_7(q), E_8(q)$, we do have the result stated in Theorem 1.2 due to the fact that we can use the result for $SL(n, q)$ for induction purpose.

Recall that CHEVIE consists of a library of data and programs to deal with the generic character table of certain finite groups G of Lie type of small rank. If G is a such group, defined over the field F_q, the generic character table of G depends on this parameter q. So, all the information stored in CHEVIE is expressed in the form of functions of q: for instance, the degrees and the number of the irreducible

characters as well as the number and the representatives of the conjugacy classes of G. See Section 5 for more details.

The CHEVIE package is better adapted to deal with groups with connected center such as $GL(n, q)$ or $CSp(2n, q)$, the conformal symplectic group. In particular, we elaborate in details the cases $GL(6, q)$ and $CSp(6, q)$. We expect that every p-vanishing character of a simple group G of Lie type in defining characteristic p is the restriction to G of a p-vanishing character of a suitable group with connected center. Unfortunately, we have not been able to prove this. Instead, we have a weaker result stating that every p-vanishing character of G is the restriction to G of a Syl_p-vanishing character of a group H with connected center (Proposition 6.2). A Syl_p-vanishing character of a finite group G means a character vanishing at all non-identity elements of a Sylow p-subgroup of G. This forces us to pay significant attention to determining Syl_p-vanishing characters of degree $|G|_p$ for groups G of small rank. In fact, we determine all Syl_p-vanishing characters of degree $|H|_p$ for quasi-simple groups H of BN-pair rank 1 and 2 and the groups $H = CSp(6, q)$, q odd, $GL(n, q)$, $n = 4, 5, 6$, $U(n, q)$, $n = 3, 4, 5$, and use certain elements of H to rule out the characters that are not p-vanishing. This gives us the list of the p-vanishing characters of degree $|H|_p$ for H. Our results on Syl_p-vanishing characters are auxiliary for the purpose of this paper, however, we expect that they will play some role in general theory.

The next problem is to perform the inductive step, and to convert information on $\tilde{\chi}$ to a certain conclusion on χ. Our tools include general machinery of Harish-Chandra induction as exposed in [7], in particular, the Benson-Curtis theorem, together with some basic theory of general Gelfand-Graev characters developed by Kotlar [17]. We define a (general) Gelfand-Graev character to be ν^G, where ν is a linear character of a Sylow p-subgroup of G for p being the defining characteristic of G. (The traditional definition of a Gelfand-Graev character (see [9, 14.29] or [3, p. 254]) requires a certain non-degeneracy condition which is dropped in our definition. Gelfand-Graev characters satisfying the non-degeneracy condition are called here non-degenerate as in [17].) Then we have $(\chi, \nu^G) = \chi(1)/|G|_p$ for every p-vanishing character χ (Lemma 2.4). This allows us to take some control of the irreducible constituents of χ that are common with ν^G for some ν.

Unfortunately, for groups other than $SL(n, q)$ there are irreducible characters that do not belong to any ν^G. We did not find any way to control such constituents of χ, and this is a source of certain difficulties. Strictly speaking, our non-computational results only concern with the "Gelfand-Graev part" $\gamma(\chi)$ of χ, where $\gamma(\chi)$ is the sum of all irreducible constituents of χ that occur in ν^G for some ν.

The formula $(\chi, \nu^G) = \chi(1)/|G|_p$ allows us to bound the multiplicities of the constituents common for χ and ν^G but does not help in identifying them. In particular, if G is with connected center of BN-pair rank 2, and if χ is p-vanishing character of degree $|G|_p$ then the number of the constituents of χ in question is at most 4, however, even in these cases we cannot avoid computations with explicit character tables. For projective module characters of degree $|G|_p$ this has been done manually, see [20] and [28]. For some groups this can be done manually in our situation too, however, some groups of rank 2 such as $^2F_4(q^2)$, seem to require unbearable volume of manual work. By using the CHEVIE package of computer programs for Chevalley groups together with their character tables available for

groups of small rank, we manage to obtain a full list of p-vanishing characters of degree $|G|_p$.

The induction step would almost be trivial if some Levi subgroup of G has no p-vanishing character of degree $|G|_p$. In fact, this never happens, as the defect zero irreducible characters of L (in particular, the Steinberg one) are p-vanishing of degree $|L|_p$. However, in some cases of interest for every maximal Levi subgroup L every p-vanishing character of degree $|G|_p$ is of defect 0. This leads to the natural question on whether it is true that a p-vanishing character is irreducible if the Harish-Chandra restriction of it to every Levi subgroup is a sum of defect 0 irreducible characters. In the case of projective indecomposable modules the Brauer-Nesbitt correspondence reduces this to a similar question for irreducible modules, which are easy to handle. But in our case no analog of the Brauer-Nesbitt correspondence is available. This explains why one cannot mimic any machinery from the theory of projective modules to deal with p-vanishing characters.

Now we continue our discussion of groups of small rank. As mentioned above, the task is to determine all Syl_p-vanishing characters of degree $|G|_p$. Technical difficulties are forced us to look for tools for simplifying computational work. We have found an advantage to determine first the characters χ of G such that $\chi|_U$ is the regular character of U, where U is a Sylow p-subgroup of G. For groups of small rank the number of such characters is not large, so it is not difficult to check which of them are p-vanishing. The matter is that the CHEVIE format of the character table is better adapted to deal with Syl_p-vanishing characters, as CHEVIE organizes them in sets such that all characters in the same set have the same restriction to U. In this way we found another helpful phenomenon of certain general interest. Specifically, we discover a considerable amount of formulas of type $\chi_1|_U = \chi_2|_U + \chi_3|_U$, where χ_i ($i = 1, 2, 3$) are irreducible characters of G. Therefore, if χ_1 occurs in a decomposition of some Syl_p-vanishing character in terms of irreducible characters, then the substitution $\chi_2 + \chi_3$ for χ_1 yields another decomposition, in which the degrees of χ_2, χ_3 are less than that of χ_1. We did not try in this paper to obtain all formulas of this kind, but those we have got are collected in the tables at the end of the paper. Partially, we expose them as tools helping the reader to control our computations. Another purpose is to provide experimental material for theoretical analysis.

Note that some formulas provided in the tables involve more than three terms (see for instance Table 11.3-B for $G = {}^2B_2(q^2)$), but they may be of interest also from this point of view.

Notation. $\mathbb{C}$ is the complex number field, $\mathbb{Z}$ is the ring of integers and F_q is the finite field of q elements. For a positive integer m we define $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$. We often view $\mathbb{Z}_m$ as the set of integers $\{1, \ldots, m\}$.

For a finite group G we denote by G', $Z(G)$ and $|G|$ the derived subgroup, the center and the order of G, respectively. If p is a prime then $|G|_p$ is the p-part of $|G|$ and also the order of every Sylow p-subgroup of G. A p'-group means a finite group with no element of order p. If $g \in G$ then $|g|$ is the order of g. All representations and modules are over $\mathbb{C}$ (unless otherwise stated). Therefore, we take liberty to use the term 'G-module'. All modules are assumed to be finitely generated. The regular representation of G is denoted by ρ_G^{reg} and the trivial one-dimensional representation is denoted by 1_G. We also use 1_G to denote the trivial

one-dimensional module and its character. The inner product of class functions η, η' is denoted by (η, η').

The set of irreducible characters of G is denoted by $\operatorname{Irr} G$, and any integral linear combination of elements of $\operatorname{Irr} G$ is called a generalized character of G.

Let H be a subgroup of G and N a normal subgroup of H. Let M be an H-module over a field. Then $C_M(N)$ is an H-module, and when it is viewed as H/N-module, it is denoted by $\overline{M}_{H/N}$ (or $\overline{M}$), and called the Harish-Chandra restriction of M to H/N. Conversely, given an H/N-module D, one can view it as an H-module with trivial action of N. Then the induced G-module D^G (when D is viewed as an H-module) is denoted by $D^{\#G}$ and called Harish-Chandra induced from D. For details see [7, p. 667, §70A], where these operations are called generalized restriction and induction. They correspond to similar operations on characters (or Brauer characters). The Harish-Chandra restriction and induction extend by linearity to class functions on G with values in $\mathbb{C}$. So if χ is a class function on G then $\overline{\chi}_{H/N}$ is the corresponding Harish-Chandra restriction of χ to H/N, and if λ is a class function on H/N then $\lambda^{\#G}$ denotes the Harish-Chandra induced class function on G. Let η be a class function on G. The formula $(\lambda^{\#G}, \eta) = (\lambda, \overline{\chi}_{H/N})$ is an easy consequence of the Frobenius reciprocity and called the Harish-Chandra reciprocity. (This is the formula 70.1(iii) in [7, p. 668].) Given a class function f on G define a function on H by $h \to \frac{1}{|N|} \sum_{n \in N} f(hn)$ ($h \in H$). This function is constant on the cosets hN and therefore defines a function on H/N called the *truncation of f to H/N* with respect to N (see [7, 70.4] or [3, pp. 262 - 283]). Sometimes we also call the truncation the function $h \to \frac{1}{|N|} \sum_{n \in N} f(hn)$ on H (which hopefully does not lead to a confusion). Note that the truncation and the Harish-Chandra restriction of a character coincide.

Let $\mathbf{H}$ be a reductive algebraic group in defining characteristic $p > 0$. An algebraic group endomorphism $Fr : \mathbf{H} \to \mathbf{H}$ is called *Frobenius* if the the subgroup $\mathbf{H}^{Fr} = \{h \in \mathbf{H} : Fr(h) = h\}$ is finite. This subgroup is called a *finite reductive group*, and p is referred as the defining characteristic of $\mathbf{H}^{Fr}$ too. Every finite reductive group is a group with BN-pair, and our use of terms 'Borel, parabolic and Levi subgroup, Weyl group' is as in the theory of groups with BN-pair [7, §65]. More detailed notation will be introduced later.

If $\mathbf{H}$ is a simple algebraic group of universal type then we refer to $\mathbf{H}^{Fr}$ as a *Chevalley group*. (Some authors additionally assume that Fr acts trivially on the Weyl group of $\mathbf{H}$, but we prefer to refer to such groups as non-twisted Chevalley groups.)

Let $H = \mathbf{H}^{Fr}$ be a finite reductive group. We use some results on the Curtis duality (called also the Curtis-Alvis duality) for the class functions on H, see [7, §71A]. If ϕ is a class function on H, we denote by $D(\phi)$ the Curtis dual of ϕ. If $D(\phi) = \pm\phi$ then we call ϕ Curtis self-dual.

Our notation for simple groups of Lie type are as in [3], however, for classical groups we also use the traditional notation such as $PSU(n, q)$ for the unitary group. Similarly, for quasi-simple groups and classical groups with connected center. For instance, $CSp(2n, q)$ denotes the conformal symplectic group.

2. p-vanishing characters and the Harish-Chandra restriction

We start with few observations valid for every finite group.

2.1. p-**vanishing and** Syl_p-**vanishing characters.** Let ϕ be a class function on a finite group G, and let p a prime dividing $|G|$. We say that ϕ is p-*vanishing* if $\phi(g) = 0$ whenever p divides $|g|$ ($g \in G$). This also applies to characters and to generalized characters. For instance, the character of the regular module is p-vanishing for every prime p. The characters of projective G-modules over a field of characteristic p are p-vanishing. It is well known that every p-vanishing function is a linear combination of the characters of projective indecomposable modules, and every p-vanishing generalized character is an integer linear combination of them, see [10, Ch. IV, Corollary 3.13] or [6, Theorem 18.26].

Furthermore, we say that ϕ is Syl_p-*vanishing* if $\phi(g) = 0$ whenever $g \neq 1$ belongs to a Sylow p-subgroup of G. Obviously, p-*vanishing* class functions are Syl_p-vanishing, but the converse is not always true. However, for irreducible characters, the converse is true (well known). (If $G = PSL(3, q)$, $q = p^k$, then there are Syl_p-vanishing reducible characters of degree $|G|_p$ that are not p-vanishing.)

If χ is a Syl_p-vanishing proper or generalized character of G then $\chi(1)$ is a multiple of $|G|_p$, and we set $c_\chi = \chi(1)/|G|_p$. This notation allows us to express certain information in a more compact form.

A proper character χ is Syl_p-vanishing if and only if the restriction of χ to a Sylow p-subgroup U of G is the character of a free U-module.

A proper p-vanishing (resp. Syl_p-vanishing) character χ is called *minimal* if whenever $\chi = \chi_1 + \chi_2$, where χ_1, χ_2 are proper characters of G, then neither χ_1 nor χ_2 is p-vanishing (respectively, Syl_p-vanishing).

Obviously, irreducible p-vanishing and Syl_p-vanishing characters are minimal and have defect 0. The converse is not true, for instance, every group $SL(2, q)$, $q = p^k$, has a reducible p-vanishing character of degree q. Thus, minimal p-vanishing characters are analogous to irreducible characters of defect 0.

Other examples of minimal p-vanishing and Syl_p-vanishing characters χ are those with $c_\chi = 1$. Note that $c_\chi = 1$ means that $\chi|_U = \rho_U^{reg}$, where U is a Sylow p-subgroup of G.

Lemma 2.1. *Let G be a finite group and let χ be a minimal p-vanishing character of G. Then all irreducible constituents of χ are in the same block.*

Proof. This follows from [10, Ch.IV, Lemma 3.14]. $\qquad\square$

Lemma 2.1 allows one to assign a p-block to every minimal p-vanishing character. Note that the lemma is not true for Syl_p-vanishing characters, even for $SL(2, q)$.

Corollary 2.2. *Let G be a perfect finite group, and let χ be a minimal p-vanishing character of G. Let $z \in Z(G)$. Suppose that $(\chi(1), |z|) = 1$. Then z belongs to the kernel of χ.*

Proof. Let ϕ be a representation of G with character χ. It follows from Lemma 2.1 that $\phi(z) = \zeta(z) \cdot \mathrm{Id}$ for some $\zeta \in \mathrm{Irr}\, Z(G)$. Then $1 = \det \phi(z) = \zeta(z)^n$, where $n = \chi(1)$. This implies $\zeta(z) = 1$. $\qquad\square$

Lemma 2.3. *Let G be a finite group with normal subgroup N, and let M be a free G-module of rank r. Then $C_M(N)$ is a free G/N-module of rank r.*

Consequently, if χ is a Syl_p-vanishing character then $\overline{\chi}_{G/N}$ is Syl_p-vanishing and $c_\chi = c_{\overline{\chi}_{G/N}}$.

Proof. The first claim is obvious for $r = 1$, and easily implies the statement. To get the second claim, apply the first one to a Sylow p-subgroup U of G in place of G and to $N \cap U$ in place of N. $\qquad\square$

Lemma 2.4. *Let G be a finite group and U a Sylow p-subgroup of G. Let η be a Syl_p-vanishing class function on G. Let τ be an irreducible character of U. Then $(\eta, \tau^G) = \eta(1)\tau(1)/|U|$, and hence $c_\eta \cdot \tau(1) = (\eta, \tau^G)$.*

Proof. We have $(\eta, \tau^G) = (\eta|_U, \tau) = \eta(1)\tau(1)/|U|$. $\qquad\square$

2.2. **The Harish-Chandra restriction.** Our standard assumption here is that G is a finite group, U a Sylow p-subgroup of G, $P \subset G$ is a subgroup containing U and $L = P/O_p(P)$. (Later on, we specify G to be a finite Chevalley group, P a parabolic subgroup of G and L a Levi subgroup of P.) Recall that for a character (or a class function) χ of G we denote by $\overline{\chi}_L$ the truncation of χ to L.

The following easy lemma is used in many situations.

Lemma 2.5. *Let G, P, L be as above.*
   (1) *Let χ be a p-vanishing (resp., Syl_p-vanishing) character of G. Then $\overline{\chi}_L$ is a p-vanishing (resp., Syl_p-vanishing) character of L and $c_\chi = c_{\overline{\chi}_L}$. (In particular, $\overline{\chi}_L \neq 0$.)*
   (2) *Let η be an irreducible constituent of $\overline{\chi}_L$. Then $(\overline{\chi}_L, \eta) = (\chi, \eta^{\#G}) > 0$.*

Proof. (1) The statement on Syl_p-vanishing characters and the equality follows from Lemma 2.3 applied to P. Assume that χ is p-vanishing. Let M be a $\mathbb{C}P$-module with character χ, and let R be the fixed point subspace of U on M. Note that $r \in R$ if and only if $r = \frac{1}{|U|}\sum_{u\in U} um$ for some $m \in M$. Let $g \in P$, $u \in U$. Suppose that the projection of g, and hence of gu, into L is not a p'-element. (Note that the projections of gu and g into L coincide.) It follows that $\overline{\chi}_L(g) = \frac{1}{|U|}\sum_{u\in U} \chi(gu) = 0$ by assumption, whence the first claim.

(2) follows from Harish-Chandra reciprocity. $\qquad\square$

Lemma 2.6. *Let G be a finite group and U a Sylow p-subgroup of G. Let χ be a Syl_p-vanishing character of G. The following are equivalent:*
   (a) $\chi(1) = |G|_p = |U|$;
   (b) $\overline{\chi}_{N_G(U)/U}(1) = 1$. *In other words, the truncation of χ over U is a linear character of $N_G(U)/U$.*

3. GELFAND-GRAEV CHARACTERS

In this section H is a finite reductive group in defining characteristic p, and U a Sylow p-subgroup of H.

Let λ be a character of U with $\lambda(1) = 1$. Then we call λ^H a *(general) Gelfand-Graev character* of H. The original Gelfand-Graev characters are called *non-degenerate* in [17], and we use this term to specify the original Gelfand-Graev characters within the set of (general) Gelfand-Graev characters. (If H is not with connected center, there can be several non-degenerate Gelfand-Graev characters [9].) Gelfand-Graev characters play a significant role in this paper. The cause of this is partially explained by the following lemma which is in fact is a special case of Lemma 2.4.

Lemma 3.1. *Let χ be a Syl_p-vanishing character of H of degree $|H|_p$. Then any Gelfand-Graev character has exactly one common irreducible constituent with χ, and this occurs with multiplicity 1 in each of them. In particular, St is a constituent of every Gelfand-Graev character.*

Let χ be a Syl_p-vanishing character. Then we can write $\chi = \gamma(\chi) + \gamma'(\chi)$, where $\gamma(\chi)$ is the sum of all irreducible constituents of χ that occur in some Gelfand-Graev character, and $\gamma'(\chi)$ is the sum of the remaining terms. Clearly, $\gamma(\chi)$ is the sum of the irreducible constituents τ of χ such that $\tau|_U$ contains a linear character of U.

Corollary 3.2. *The number of irreducible constituents of $\gamma(\chi)$ does not exceed the number of distinct Gelfand-Graev characters.*

Lemma 3.3. *Let χ be a Syl_p-vanishing character of H of degree $|H|_p$. Then $\gamma(\chi)$ is multiplicity free.*

Proof. This follows from the equality $\chi|_U = \rho_U^{reg}$, and is also an immediate consequence of Lemma 3.1. $\square$

Remark 3.4. The character γ' is not always multiplicity free. For instance, for $H = PSU(3,5)$, in notation of Atlas [4], the character $\chi_1 + 2\chi_2 + \chi_4 + \chi_5 + \chi_6$ is 5-vanishing of degree $|H|_5 = 125$.

The Gelfand-Graev characters can be characterized as follows:

Lemma 3.5 ([17, Lemma 2.3])**.** *Suppose that H is with connected center, with no component of type $B_n(2), C_n(2), F_4(2), G_2(2), G_2(3), {}^2B_2(2), {}^2F_4(2), {}^2G_2(3)$. Let λ be a linear character of U. Then $\lambda^H = \Gamma_L^{\#H}$, where L is some Levi subgroup of H and Γ_L is the non-degenerate Gelfand-Graev character of L. Conversely, if Γ_L is the non-degenerate Gelfand-Graev character of L, then $\Gamma_L^{\#H} = \lambda^H$ for some linear character λ of U.*

Remark. The exclusions in the above lemma correspond to the cases where $U \neq U_1$ in notation of [17, p. 350]. The statement in [17] does not exclude the groups H with components of type ${}^2B_2(2), {}^2F_4(2), {}^2G_2(3)$, however, we think that this is required.

Let $\mu \in \operatorname{Irr} H$ and let L be a Levi subgroup of H. Suppose that μ is not a constituent of any Gelfand-Graev character of H. Then all irreducible constituents of $\overline{\mu}_L$ belong to $\gamma'(\overline{\chi}_L)$. Indeed, suppose the contrary, and let λ be an irreducible constituent of $\overline{\mu}_L$ belonging to $\gamma(\overline{\chi}_L)$. This means that λ is a constituent of $\Gamma_K^{\#L}$ for some Levi subgroup K of L (and hence of H) and a non-degenerate Gelfand-Graev character Γ_K of K. Note that $(\mu, \lambda^{\#H}) = (\overline{\mu}_L, \lambda) > 0$ by Harish-Chandra reciprocity. Therefore, μ is a constituent of $\Gamma_K^{\#H}$ by transitivity of Harish-Chandra induction. Hence, by Lemma 3.5, we get a contradiction.

Note that a regular character σ is cuspidal if and only if σ is semisimple. Indeed, consider the Curtis dual $D(\sigma)$ of σ. As σ is regular, $\varepsilon D(\sigma)$ is semisimple for some $\varepsilon = \pm 1$ [9, 14.39]. However, σ is cuspidal if and only if $D(\sigma) = \pm\sigma$ [7, p. 690], so the claim follows.

Lemma 3.6. *Let H be as in Lemma 3.5. Then the number of distinct Gelfand-Graev characters equals the number of non-conjugate Levi subgroups of H.*

Proof. By Lemma 3.5, every Gelfand-Graev character is of shape $\Gamma_L^{\#H}$, where Γ_L is the non-degenerate Gelfand-Graev character of L. (Here L can coincide with H.) Since H is with connected center then so is L (see [3, 8.1.4]), and hence L has a unique non-degenerate Gelfand-Graev character. Clearly, if Levi subgroups L, M of H are conjugate then $\Gamma_L^{\#H}$ coincides with $\Gamma_M^{\#H}$. (Note that $\Gamma_L^{\#H}$ does not depend on the choice of the parabolic subgroup which defines L as a Levi subgroup [7, 70.10].) Suppose that L, M are not conjugate. It is known that each L, M has a cuspidal regular character η_L, η_M, say. (Indeed, if a maximal torus T^* of H^* contains an element s in general position then the regular character ρ_s corresponding to s is cuspidal if and only if T^* contained in no proper Levi subgroup of H^* (see [3, 9.3.2]). It is shown by Veldkamp [27, p. 391] that a so called Coxeter torus T^* is not contained in any proper Levi subgroup of H^* and contains an element in general position; it is also known that an element in general position is contained in a unique maximal torus [27, p. 390].) Then $\eta_L^{\#H}$, $\eta_M^{\#H}$ are disjoint, and their irreducible constituents are contained in $\Gamma_L^{\#H}$, $\Gamma_M^{\#H}$, respectively. So $\Gamma_L^{\#H} \neq \Gamma_M^{\#H}$, and the result follows. $\qquad\square$

Remark. One observes that this result is not valid for $H = {}^2B_2(2), {}^2G_2(3), {}^2F_4(2)$.

The following result is contained in [17, Propositions 2.4, 2.5, Corollary 2.6].

Lemma 3.7. *Let H be as in Lemma 3.5, and $\tau \in \operatorname{Irr} H$. Then the following are equivalent:*

(1) *τ is a constituent of some Gelfand-Graev character of H;*
(2) *τ is a constituent of $\lambda^{\#H}$, where λ is a cuspidal regular character of a Levi subgroup of some parabolic subgroup of H.*

Observe that the character λ in Lemma 3.7 is both regular and semisimple because every cuspidal regular character is semisimple (see comments prior Lemma 3.6).

Distinct Gelfand-Graev characters have common irreducible constituents. Recall that a *Harish-Chandra series of H* is the set of irreducible constituents of a Harish-Chandra induced character $\lambda^{\#H}$, where L is a Levi subgroup of a parabolic subgroup P of H (possibly $P = H$) and λ is a cuspidal character of L. Two Harish-Chandra series either coincide or are disjoint, see [7, 70.15A]. Therefore, if λ is cuspidal then the irreducible constituents of $\lambda^{\#H}$ do not occur in the other Gelfand-Graev characters.

In addition, if σ is a regular cuspidal character of L then the irreducible constituents of $\sigma^{\#H}$ are in bijection with $\operatorname{Irr} W_\sigma$, where W_σ is a certain subgroup of W generated by reflections. More precisely, W acts on the roots of the corresponding algebraic group, and L is defined by a subset J of simple roots. Then $W = \{w \in W : w(J) = J \text{ and } w(\sigma) = \sigma\}$.

Proposition 3.8. *Assume H to be with connected center and has no component of type $B_n(2)$, $C_n(2)$, ${}^2B_2(2)$, $F_4(2)$, ${}^2F_4(2)$, $G_2(2)$, $G_2(3)$ and ${}^2G_2(3)$. Let χ be a Syl_p-vanishing character of H of degree $|H|_p$.*

(1) *For every irreducible constituent of τ of $\gamma(\chi)$ there are unique Levi subgroup L of H and unique cuspidal regular character ρ of L such that $(\tau, \rho^{\#H}) > 0$. Moreover, $(\tau, \rho^{\#H}) = 1$.*

(2) Let $l(\chi)$ be the number of non-conjugate Levi subgroups of H such that $\overline{\chi}_L$ contains a cuspidal regular character of L. Then the number of irreducible constituents of $\gamma(\chi)$ equals $l(\chi)$.

Proof. Let $\{\nu_1, \ldots, \nu_k\}$ be a maximal set of linear characters of U such that $\nu_i^H \neq \nu_j^H$ for $i \neq j$. By Lemma 3.5, $\nu_i^H = \Gamma_L^{\#H}$ for some Levi subgroup L, and by Lemma 3.6, the set $\{\nu_1, \ldots, \nu_k\}$ is in bijection with the set of non-conjugate Levi subgroups. Let L_i correspond to ν_i for $i = 1, \ldots, k$. As $(\chi, \nu_i^H) = 1$ (Lemma 3.1), we may denote by χ_i the constituent of χ common with ν_i^H. (So χ_i is a constituent of $\gamma(\chi)$.) The mapping $i \to \chi_i$ is surjective, but not injective in general.

To see when $\chi_i = \chi_j$, fix some j. Then χ_j is a constituent of $\Gamma_{L_j}^{\#H}$, so there is a regular character ρ_j of L_j such that $(\chi, \rho_j^{\#H}) = 1$. Moreover, ρ_j is unique. Indeed, as $\overline{\chi}_{L_j}(1) = |L_j|_p$ and $\overline{\chi}_{L_j}$ is Syl_p-vanishing, we have $(\overline{\chi}_{L_j}, \Gamma_{L_j}) = 1$, where Γ_{L_j} is a non-degenerate Gelfand-Graev character of L_j. As L_j is with connected center too [3, 8.1.4], Γ_{L_j} is unique, and hence ρ_j is unique.

Suppose that ρ_j is not cuspidal. Then ρ_j is a constituent of $\lambda_i^{\#L_j}$, where L_i is some Levi contained in L_j (up to conjugation) and λ_i is a cuspidal regular character of L_i (Lemma 3.7). Therefore, χ_j is a constituent of ν_i^H too, and hence we have $\chi_i = \chi_j$ (as $(\chi, \nu_i^H) = 1$). Thus we need not count ν_j such that $\overline{\chi}_{L_j}$ contains no cuspidal regular character of L_j, and the claim follows.   □

Lemma 3.9. *Let H be as in Proposition* 3.8 *and χ a Syl_p-vanishing character of H. Suppose that $\gamma'(\overline{\chi}_L) = 0$ for every proper Levi subgroup L of H. Then every irreducible constituent of $\gamma'(\chi)$ is a cuspidal character.*

Proof. Suppose the contrary. Let σ be an irreducible constituent of $\gamma'(\chi)$, which is not cuspidal. Then there exists a Levi subgroup L and a cuspidal character $\mu \in \operatorname{Irr} L$ such that $(\sigma, \mu^{\#H}) > 0$. Therefore, $(\overline{\sigma}_L, \mu) = (\sigma, \mu^{\#H}) > 0$. So μ is an irreducible constituent of $\overline{\sigma}_L$, and hence of $\overline{\chi}_L$. By assumption, $\gamma'(\overline{\chi}_L) = 0$. This means that μ is a constituent of a Gelfand-Graev character of L. By Lemma 3.7, μ is a regular character of L (as μ is cuspidal). (In fact, we use induction on the BN-pair rank of H.) Therefore, μ is a constituent of Γ_L, the non-degenerate Gelfand-Graev character of L. By Lemma 3.6, $\Gamma_L^{\#H}$ is a Gelfand-Graev character of H. As σ is a constituent of $\mu^{\#H}$, it is also a constituent of $\Gamma_L^{\#H}$, that is, σ is a summand of $\gamma(\chi)$, which is a contradiction.   □

Lemma 3.10. *Let H be a finite reductive group and let τ, σ be distinct irreducible constituents of $\gamma(\chi)$ for a Syl_p-vanishing character χ of degree $|H|_p$. Then $\tau|_U$ and $\sigma|_U$ have no common constituent of degree 1, in particular, $\tau|_U \neq \sigma|_U$.*

Proof. Indeed, suppose the contrary. Then $(\tau|_U, \nu) > 0$ and $(\sigma|_U, \nu) > 0$ for some linear character ν of U. As $\chi|_U = \rho_U^{reg}$, we have $(\chi|_U, \nu) = 1$, and the claim follows.   □

Lemma 3.11. *Let $H = GL(n, q)$ or $SL(n, q)$. Then every irreducible character of H is a constituent of a Gelfand-Graev character.*

Proof. For $H = GL(n, q)$ the result is stated in the original paper by Gelfand and Graev [13], and this implies the one for $G = H' = SL(n, q)$. Indeed, let U be a Sylow p-subgroup of G. Let $\lambda \in \operatorname{Irr} U$ with $\lambda(1) = 1$. By Mackey's lemma, $(\lambda^H)|_G = \sum_g (\lambda^g)^G$, where g are the representatives of the double cosets of G and

U in H. Since $H = G \cdot T$, where T is a maximal torus of the Borel subgroup of H containing U [19, Lemma 24.12], we may choose g in T. Then U is g-stable, λ^g is a linear character of U and $(\lambda^g)^G$ is a Gelfand-Graev character of G. So the result follows from that for $GL(n, q)$, as every irreducible character of G is a constituent of the restriction to G of an irreducible character of H. $\qquad\square$

Corollary 3.12. *Let $H = GL_n(q)$ or $SL_n(q)$, and let χ be a Syl_p-vanishing character of H of degree $|H|_p$. Then $\chi = \gamma(\chi)$, and hence χ is multiplicity free.*

Proof. This follows from Lemmas 3.11 and 3.3. $\qquad\square$

4. Syl_p-DECOMPOSABLE IRREDUCIBLE CHARACTERS

Let τ be an irreducible character of a finite group G. Let U be a Sylow p-subgroup of G. We say that τ is Syl_p-decomposable if there are irreducible characters $\chi_1, \ldots, \chi_k$ of G (not necessarily distinct) such that $\tau(u) = \chi_1(u) + \cdots + \chi_k(u)$ for every element $u \in U$. (Equivalently, $\tau - \chi_1 - \cdots - \chi_k$ vanishes on U.) We express this by writing $\tau \equiv \chi_1 + \cdots + \chi_k \pmod{U}$, and call this expression a Syl_p-*decomposition* of τ. Note that some characters of G can be Syl_p-*equivalent*, that is, they take the same value at every $u \in U$.

The problem of determining the Syl_p-vanishing characters of degree $|U|$ discussed above for finite reductive groups in defining characteristic p can be restated as that of finding all Syl_p-decompositions of the Steinberg character of G. We find that the knowledge of Syl_p-decompositions for characters τ of degree less that $|U|$ is very helpful for determining the Syl_p-vanishing characters. Indeed, if τ is a constituent of a Syl_p-vanishing character χ then the substitution of $\sum \chi_i$ for τ yields a new Syl_p-vanishing character, and the degree of new irreducible constituents are less than $\tau(1)$. In addition to this, we have found out quite a lot of Syl_p-decomposable characters, including regular and semisimple, and this hints that the use of them can organise Syl_p-vanishing characters in a more conceptual way.

In this section we collect some general facts on Syl_p-decompositions for finite reductive groups H.

Recall that, if ϕ is a class function on H then $D(\phi)$ denotes the Curtis dual of ϕ. If $D(\phi) = \pm\phi$ then we call ϕ Curtis self-dual. The following well known fact can be easily deduced from the definitions of the duality operation [7, 71.2] and the Steinberg character [7, 71.18].

Lemma 4.1. *Let ϕ be a class function on H constant on U. Then $D(\phi)(g) = St(g)\phi(g)$ for all $g \in H$. In particular, if $\phi(u) = 0$ for all $u \in U$ then $D(\phi)(u) = 0$ for all $u \in U$.*

Lemma 4.2. *Suppose H is with connected center. Let $\chi_1, \chi_2, \chi_3 \in \operatorname{Irr} H$. Suppose that $\chi_1 \equiv \chi_2 + \chi_3 \pmod{U}$ is a Syl_p-decomposition of χ_1.*

    (1) *At least one of the characters χ_i, $1 \leq i \leq 3$, is not regular.*

    (2) *At least one of the characters χ_i, $1 \leq i \leq 3$, is not semisimple.*

    (3) *If every χ_i is either regular or semisimple then at least one of them is regular and semisimple (and hence Curtis self-dual).*

    (4) *$D(\chi_1) - D(\chi_2) - D(\chi_3)$ is a Syl_p-vanishing generalized character of degree 0. Let $\chi_i' = \pm D(\chi_i)$ be a proper character, $i = 1, 2, 3$. Then reordering of the χ_i' yields a Syl_p-decomposition (which may coincide with that in the assumption).*

Proof. Note that $\chi_1 - \chi_2 - \chi_3$ is a Syl_p-vanishing generalized character of degree 0. By Lemma 4.1, so is $D(\chi_1 - \chi_2 - \chi_3)$, which implies (4). Let Γ be the non-degenerate Gelfand-Graev character of H.

(1) If all χ_i are regular, then, taking the inner product of $\chi_1 - \chi_2 - \chi_3$ with Γ, we get $1 - 1 - 1 = 0$, which is absurd.

(2) Suppose all of them are semisimple. As $\pm D(\chi_i)$ is a regular character for $i = 1, 2, 3$, the inner product with Γ yields again $\pm 1 \pm 1 \pm 1 = 0$, a contradiction.

(3) Suppose that none of the χ_i's is regular and semisimple simultaneously. If one of the characters is regular and the other two are not regular, the inner product with Γ yields a contradiction. So there are two regular characters among them, and they are not semisimple. Then $D(\chi_1) - D(\chi_2) - D(\chi_3)$ is Syl_p-vanishing, and two characters among $D(\chi_1), -D(\chi_2), -D(\chi_3)$ are semisimple and not regular, and one is regular. Again, the inner product with Γ yields a contradiction. $\square$

Lemma 4.3. *Suppose H is with connected center. Let χ be a Syl_p-vanishing character of H of degree $|H|_p$, and let σ be a regular character of H that occurs as a constituent of $\gamma(\chi)$. Let $\tau = \sum m_i \tau_i$ be a reducible character such that $\sigma - \tau$ is Syl_p-vanishing of degree 0.*

(1) *There is a unique i such that τ_i is regular;*

(2) *Suppose H has no component of type $B_n(2), C_n(2), {}^2B_2(2), F_4(2), {}^2F_4(2), G_2(2), G_2(3), {}^2G_2(3)$. If σ is cuspidal then $\gamma(\tau) = \tau_i$, that is, τ_j does not belong to any Gelfand-Graev character for $j \neq i$; in addition, if $H = GL(n, q)$ then τ is irreducible and hence Syl_p-equivalent to σ;*

(3) *$\gamma(\tau)$ is multiplicity free, that is, $m_i = 1$ if τ_i is a constituent of some Gelfand-Graev character.*

Proof. (1) follows by taking the inner product of $\sigma - \tau$ with the non-degenerate Gelfand-Graev character of H.

(2) By the way of contradiction, suppose that τ_j for $j \neq i$ is a constituent of a Gelfand-Graev character Γ of H. By Lemma 3.5, there is a Levi subgroup L of H such that $\Gamma = \Gamma_L^{\#H}$. Note that $L \neq H$, as $(\tau, \Gamma) = (\tau_i, \Gamma) = 1$. Since $(\tau_j, \Gamma_L^{\#H}) > 0$, we have $(\sigma, \Gamma_L^{\#H}) > 0$, which is false as σ is cuspidal.

To prove the additional statement, recall that every irreducible character of $H = GL(n, q)$ is a constituent of some Gelfand-Graev character (Lemma 3.11). This implies $\gamma(\tau) = \tau$.

(3) Let τ_i is a constituent of a Gelfand-Graev character ν^H. Suppose that $m_i > 1$. Then $(\tau, \nu^H) \geq m_i > 1$, and hence $(\sigma, \nu^H) > 1$. This implies $(\chi, \nu^H) > 1$, which contradicts Lemma 3.1. $\square$

Lemma 4.4. *Let $\chi, \chi_1, \ldots, \chi_k \in \mathrm{Irr}\, H$, and $\chi(1) < |H|_p$. Suppose that $\chi(g) = \chi_1(g) + \cdots + \chi_k(g)$ for every p-singular element $g \in H$. Then all characters $\chi, \chi_1, \ldots, \chi_k$ belong to the same p-block.*

Proof. Let B be the block χ belongs to. By reordering the χ_i's we can assume that $\chi_1, \ldots, \chi_m \in B$, and $\chi_{m+1}, \ldots, \chi_k \notin B$. By Feit [10, Ch. IV, Lemma 3.14], it follows that $\sigma := \chi_{m+1} + \cdots + \chi_k$ is a p-vanishing character, and hence $\sigma(1)$ is a multiple of $|H|_p$. So $\sigma(1) = 0$, and hence $\sigma = 0$, as claimed. $\square$

The use of Syl_p-decompositions of irreducible characters for computations of the Syl_p-vanishing characters of degree of $|H|_p$ is very helpful and seems to be unavoidable for most groups of rank greater than 1. However, we feel that they

are of certain independent interest. Because of these we perform an extensive computations of them, and collect the relevant results in the tables at the end of the paper.

5. Chevalley groups of BN-pair rank 1

In this section as well as in Sections 7 and 9, we compute the Syl_p-vanishing characters ψ of degree $|G|_p$ for some Chevalley groups G, whose character tables are known. Our main reference here is the algebra package CHEVIE [12], in particular, its MAPLE part. We keep the notation of the CHEVIE package for the irreducible characters and the conjugacy classes. In particular, CHEVIE partitions $\mathrm{Irr}\, G$ in sets, which we call CHEVIE *sets* and denote by $\mathbf{X}_h = \{\chi_h(k) \mid k \in I_h\}$ in this paper. Here h is a natural number and I_h is some parameter set, called the character parameter group (see the CHEVIE manual [12]).

Furthermore, CHEVIE partitions the conjugacy classes of G in sets $\mathbf{C}_h = \{C_h(k) \mid k \in I_h\}$, where, as before, I_h is the corresponding parameter set. Note the classes belonging to the same set $\mathbf{C}_h$ have the same order.

To avoid misunderstanding, we emphasize that CHEVIE notation for parameters $k \in I_h$ for different groups is often not homogeneous, but we are forced to follow CHEVIE. For instance, the set I_4 of $U(3,q)$ is parametrised by (u,v), and the set I_{10} of $SU(3,q)$, with $3 \mid (q+1)$, is parametrised by (n,m).

It is essential for us that the characters in the same CHEVIE set $\mathbf{X}_h$ are Syl_p-equivalent (but the converse is not true). This allows us, in performing computations with Syl_p-vanishing characters, to ignore the parameter k that determines an individual character $\chi_h(k) \in \mathbf{X}_h$. By this reason we write χ_h to denote any character of $\mathbf{X}_h$ (to simplify the notation).

In order to facilitate computations, we first compute some Syl_p-decompositions of the irreducible characters χ_h of G, that is, the equalities that are true for every $u \in U$:

$$(5.1) \qquad\qquad \chi_h \equiv \chi_{j_1} + \ldots + \chi_{j_t} \pmod{U}.$$

The results are given in the tables of Section 11. For Syl_p-decompositions we choose to compact the notation by writing $\mathbf{v} = (j_1, \cdots, j_t)$, especially in the tables. The case where $t = 1$ means that χ_h and χ_{j_1} are Syl_p-equivalent. (This is reflected in the tables, see, for instance Table 11.2.3-B.)

Our strategy involves three steps. First, we determine Syl_p-vanishing characters of degree $|G|_p$ whose irreducible constituents χ_j has no Syl_p-decompositions (with $t > 1$). Next we use the Syl_p-decompositions to obtain all Syl_p-vanishing characters in question by substituting χ_h for $\chi_{j_1} + \cdots + \chi_{j_t}$ whenever it is possible. Finally, we use the character table of G in order to detect and rule out those Syl_p-vanishing characters that are not p-vanishing. In most cases we do this for groups with connected center. However, in the last step we try to only use elements that belong to the derived subgroup G' of G. In view of Proposition 6.2, this yields the result for groups with disconnected center.

By Lemma 3.1, every Syl_p-vanishing character ψ of degree $|G|_p$ has at least one regular character constituent. We use CHEVIE to identify the regular characters of G in our list of Syl_p-decompositions. This allows us to assume that every regular constituent of ψ is Syl_p-indecomposable. We implicitly use this procedure for each group in the tables of Section 11 without references.

Note that every character $\chi_h \in \mathbf{X}_h$ occurs with the same multiplicity in a Gelfand-Graev character ν^G for a fixed $\nu \in \operatorname{Irr} U$. In particular, by Lemma 2.4, we have:

Lemma 5.1. *Suppose that G is with connected center. If some character $\chi_h \in \mathbf{X}_h$ is regular then so are all characters in $\mathbf{X}_h$. In addition, if χ is a Syl_p-vanishing character of G of degree $|G|_p$ then for any fixed h there is at most one regular character from $\mathbf{X}_h$ occurring in χ.*

Below we often use the following simple observation. Let H be a finite reductive group and G the subgroup generated by unipotent elements. Then H/G is abelian. Suppose that G is quasi-simple. Let χ be a p-vanishing character of H of degree $|H|_p$. Then χ contains a one-dimensional constituent if and only if $\chi|_G$ contains 1_G. Therefore, there is a one-dimensional character ν of H such that $\chi \cdot \nu$ contains 1_H.

In this section, as well as in Sections 7 and 9, we shall use the following notation for certain complex numbers:

$$\omega = \exp(\tfrac{2\pi i}{3}), \qquad \zeta_1 = \exp(\tfrac{2\pi i}{q-1}), \qquad \xi_1 = \exp(\tfrac{2\pi i}{q+1}), \qquad \xi_2 = \exp(\tfrac{2\pi i}{q^2+1}),$$
$$\xi_3 = \exp(\tfrac{2\pi i}{q^3+1}), \qquad \varphi_8' = \exp(\tfrac{2\pi i}{q^2+\sqrt{2}q+1}), \qquad \varphi_{12}'' = \exp(\tfrac{2\pi i}{q^2+\sqrt{3}q+1}).$$

Now we start dealing with Chevalley groups of rank 1: $SL(2,q)$, $SU(3,q)$, ${}^2B_2(q^2) = Sz(q^2)$ with $q^2 = 2^{2m+1}$, and ${}^2G_2(q^2)$ with $q^2 = 3^{2m+1}$.

5.1. **Groups $SL(2,q)$ and $GL(2,q)$.** One observes that all irreducible characters of $GL(2,q)$ of the same degree are Syl_p-equivalent. Inspection of the character table of $G = SL(2,q)$ yields:

Lemma 5.2. *Let $G = SL(2,q)$ and ψ be a reducible character of degree q. Then ψ is Syl_p-vanishing if and only if one of the following holds:*

 (1) *$\psi - 1_G$ is an irreducible character;*
 (2) *$\psi - 1_G = \eta_1 + \eta_2$, where η_1, η_2 are distinct characters of degree $(q-1)/2$;*
 (3) *$\psi = \tau + \eta$, where $\tau(1) = (q+1)/2$ and $\eta(1) = (q-1)/2$ and τ, η belong to distinct Weil representations of G.*

In addition, ψ is p-vanishing if and only if ψ is Syl_p-vanishing and $\psi(z) = \psi(1)$ for all $z \in Z(G)$.

Remarks 5.3. (i) The character $\psi - 1_G$ in (1) and η_1, η_2 are cuspidal. (ii) The cases (2), (3) occur only for q odd. (iii) The cases (3) cannot occur for $G = PSL(2,q)$. (iv) In (3) the characters τ, η are in distinct blocks.

Lemma 5.4. *Let $H = GL(2,q)$. A reducible character ψ of H of degree q is Syl_p-vanishing if and only if $\psi = \lambda + \eta$, where $\lambda(1) = 1$, and η is an irreducible cuspidal character of degree $\eta(1) = q - 1$. In this case, ψ is p-vanishing if and only if $Ker(\psi) = Z(H)$.*

In CHEVIE notation, $\psi = \chi_1(k_1) \cdot (1_H + \chi_4(k_4))$, where $\chi_1(k_1)$ $(k_1 \in I_1)$ is a linear character of H and $k_4 \in I_4$ with $k_4 = (q-1)k_4'$ (see Tables 11.1-A,B,C,D).

5.2. **The unitary groups** $SU(3,q)$ **and** $U(3,q)$**.** We first consider the groups $H = U(3,q)$. According to CHEVIE, we partition Irr H into 8 sets $\mathbf{X}_h$, and the characters in $\mathbf{X}_h$ are parametrized by a corresponding set I_h of parameters, see Table 11.2.1-A. Due to the Syl_p-decompositions of Table 11.2.1-B, we first assume that $h \in \{1,2,6\}$. Using the character table of H, it is immediate to obtain a unique Syl_p-vanishing character of degree $|H|_p$ whose constituents belong to these $\mathbf{X}_h$: $\psi_1 = \chi_1 + \chi_2 + \chi_2 + \chi_6$. Now, applying to this character the Syl_p-decompositions of Table 11.2.1-B, we get all Syl_p-vanishing characters of H of degree $|H|_p$ (see Table 11.2.1-C).

To determine the p-vanishing characters of H of degree $|H|_p$, we use the character table of H. For the purpose of illustration, here and in some cases below, we provide the details.

Lemma 5.5. *Let $H = U(3,q)$. Then H admits a reducible p-vanishing character ψ of degree $|H|_p$ if and only if $3 \mid (q+1)$. In this case, there are exactly $q+1$ characters ψ, all of type*

$$\psi = \chi_1(u) \cdot (1_H + \chi_2(a) + \chi_2(2a) + \chi_6(a, 2a, 3a)),$$

where $a = (q+1)/3$ and $\chi_1(u)$ is a linear character of H.

Proof. Let ψ be a Syl_p-vanishing character of H of degree $|H|_p = q^3$. By Lemma 2.1, all the constituents of ψ belong to the same block. So it suffices to look at the classes $C_5(k,l)$, where $(k,l) \in \mathbb{Z}_{q+1} \times \mathbb{Z}_{q+1}$ and $k \neq l$. We have one of the following cases, modulo linear characters of H:

(1) Take $\psi_1 = 1_H + \chi_2(u_2) + \chi_2(u_2') + \chi_6(u_6, v_6, w_6)$. As all the summands are in the same block, it follows that $q+1$ divides $3u_2$, $3u_2'$ and $u_6 + v_6 + w_6$. We obtain

$$1 + \xi_1^{-(k-l)u_2} + \xi_1^{-(k-l)u_2'} = \xi_1^{(k-l)(u_6+v_6)} + \xi_1^{-(k-l)v_6} + \xi_1^{-(k-l)u_6}.$$

This holds for every k,l with $k \neq l$. This implies that 3 divides $q+1$, and, by symmetry, $u_2 = a$, $u_2' = 2a$ and $(u_6, v_6, w_6) = (a, 2a, 3a)$, with $a = (q+1)/3$.

(2) Take $\psi_2 = \chi_2(0) + \chi_4(u_4, v_4) + \chi_6(u_6, v_6, w_6)$, with $q+1$ dividing both $2u_4 + v_4$ and $u_6 + v_6 + w_6$. We obtain

$$1 + \xi_1^{-(k-l)u_4} + \xi_1^{2(k-l)u_4} = \xi_1^{(k-l)(u_6+v_6)} + \xi_1^{-(k-l)v_6} + \xi_1^{-(k-l)u_6}.$$

We have the following possibilities:
   (a) Either $u_6 = 0$ or $v_6 = 0$: this implies $u_4 = 0$ and $v_4 = 0$, a contradiction with the conditions on the parameters.
   (b) $u_6 + v_6 = 0$: again we obtain $u_4 = v_4 = 0$.
(3) Take $\psi_3 = \chi_2(0) + \chi_5(u_5, v_5)$: we obtain $\xi_1^{k(u_5+v_5)+lu_5} = 1$, which implies $u_5 = v_5 = 0$, a contradiction with the conditions on the parameters.

$\square$

Next, let $G = SU(3,q)$, where $3 \nmid (q+1)$. This case is very similar to the case $U(3,q)$. We list the the relevant informations in Tables 11.2.2-A,B,C. Thus, it can be easily proved the following

Lemma 5.6. *Let $G = SU(3,q)$, where $3 \nmid (q+1)$. Then G admit no reducible p-vanishing character of degree $|G|_p$.*

Finally, let $G = SU(3, q)$ with $3 \mid (q+1)$. We partition $\operatorname{Irr} G$ into 17 CHEVIE sets, see Table 11.2.3-A. The Syl_p-decompositions of Table 11.2.3-B allow us to work only with the characters χ_h where $h \in \{1, 2, 7, 8, 9\}$. So, using the character table, it is immediate to obtain the Syl_p-vanishing character $\psi_1 = \chi_1 + 2\chi_2 + \chi_7 + \chi_8 + \chi_9$. Next, applying the Syl_p-decompositions of Table 11.2.3-B to this character, we are able to list all the Syl_p-vanishing characters of G of degree $|G|_p$ (see Table 11.2.3-C).

Lemma 5.7. *Let $G = SU(3, q)$, with $3 \mid (q+1)$. Then G admits a unique reducible p-vanishing character ψ of degree $|G|_p$, namely*

$$\psi = 1_G + 2\chi_2 + \chi_7 + \chi_8 + \chi_9.$$

Proof. The cases $q = 2, 5, 8$ can be easily dealt with MAGMA. So, assume $q \geq 11$. Applying Lemma 2.1, we easily rule out all the options where the irreducible constituents of ψ are not in the same block. So, we have to study only the following five cases:

(1) $\psi_1 = \chi_1 + 2\chi_2 + \chi_7 + \chi_8 + \chi_9$: this character is p-vanishing.
(2) $\psi_2 = \chi_1 + 2\chi_2 + \chi_{10}(n, m)$: on the class $C_7(1)$ we obtain $\xi_1^{n-2m} + \xi_1^{m-2n} + \xi_1^{n+m} = 3$. So, $\xi_1^{n-2m} = \xi_1^{m-2n} = \xi_1^{n+m} = 1$ and hence $(m, n) = (\frac{q+1}{3}, \frac{2(q+1)}{3})$, $(\frac{2(q+1)}{3}, \frac{q+1}{3})$. But, for this choice of the parameters, $\chi_{10} = \chi_7 + \chi_8 + \chi_9$.
(3) $\psi_{21} = \chi_2 + \chi_4(k) + \chi_7 + \chi_8 + \chi_9$: on the class $C_7(1)$ we obtain $\xi_1^k + \xi^{-2k} = 2$ and so $\xi_1^k = 1$, a contradiction.
(4) $\psi_{22} = \chi_2 + \chi_4(k) + \chi_{10}(n, m)$: on the classes $C_7(a)$ $(a = 1, 2, 3)$ we obtain $1 + \xi_1^{ak} + \xi_1^{-2ak} = \xi_1^{a(n-2m)} + \xi_1^{a(m-2n)} + \xi_1^{a(n+m)}$. It follows that $(m, n) = (\frac{q+1}{3}, \frac{2(q+1)}{3}), (\frac{2(q+1)}{3}, \frac{q+1}{3})$. So $(q + 1) \mid k$, a contradiction.
(5) $\psi_{23} = \chi_2 + \chi_5(n)$: on the class $C_7(1)$ we obtain $\xi_1^n = 1$, i.e. $(q + 1) \mid n$, a contradiction with the conditions on the parameter set I_5.

Thus, ψ_1 is the only reducible p-vanishing character of G. $\qquad\square$

5.3. Groups $^2B_2(q^2)$. Let $G = {}^2B_2(q^2)$, with $q^2 = 2^{2m+1}$, $m \geq 0$. According to CHEVIE, we partition $\operatorname{Irr} G$ into 7 sets $\mathbf{X}_h$, see Table 11.3-A. The Syl_p-decompositions of Table 11.3-B allow us to work only with the characters χ_h for $h \in \{1, 2, 3, 6\}$. Let ψ be a Syl_p-vanishing character whose constituents are χ_h with $h \in \{1, 2, 3, 6\}$. It is immediate to obtain a unique Syl_p-vanishing character of degree $|G|_p$ with these constituents: $\psi_1 = \chi_1 + \chi_2 + \chi_3 + \chi_6$. Now, applying the Syl_p-decompositions of Table 11.3-B to ψ_1, we are able to list all the Syl_p-vanishing characters of G of degree $|G|_p$ (see Table 11.3-C). As every element of G is either semisimple or unipotent, every Syl_p-vanishing character is p-vanishing. Thus, we obtain the following.

Lemma 5.8. *Let $G = {}^2B_2(q^2)$, $q^2 = 2^{2m+1}$. Then G admits exactly $2^{m-1}(2^m + 1)$ reducible p-vanishing characters ψ of degree $|G|_p$, all of type*

$$\psi = 1_G + \chi_2 + \chi_3 + \chi_6(k).$$

5.4. Groups $^2G_2(q^2)$. Let $G = {}^2G_2(q^2)$, with $q^2 = 3^{2m+1}$, $m \geq 0$. We partition $\operatorname{Irr} G$ into 14 sets $\mathbf{X}_h$, see Table 11.4-A. In order to reduce the number of the characters χ_h we have to consider, it is easier to work first only with the classes C_1, C_2, C_5. We obtain the decompositions listed in Table 11.4-B. These decompositions allow us to work only with the characters χ_h, where $h \in \{1, 4, 5, 7, 9, 12, 13, 14\}$. Working only with these constituents, we obtain a unique character of degree $|G|_p$ vanishing

on the classes C_2 and C_5: $\theta = \chi_1 + 2\chi_5 + 2\chi_7 + \chi_{14}$. Now, we apply the decompositions of Table 11.4-B to the character θ, to verify if the resulting character is Syl_p-vanishing. At the end, we are able to list all Syl_p-vanishing characters of G of degree $|G|_p$ (see Table 11.4-C).

Lemma 5.9. *Let $G = {}^2G_2(q^2)$, with $q^2 = 3^{2m+1}$. Then G admits exactly $3^m(3^m + 1)/2$ reducible p-vanishing characters ψ of degree $|G|_p$, all of type*

$$\psi = 1_G + \chi_3 + \chi_5 + \chi_6 + \chi_7 + \chi_{14}(k).$$

Proof. Direct inspection of the character table shows that every Syl_p-vanishing character of G of degree $|G|_p$ is p-vanishing.   □

Observe that the group $G = {}^2G_2(3)'$ has a unique reducible 3-vanishing character of degree $|G|_3$: $\phi = \chi_1 + \chi_6$ (with the notation of [4]). On the other hand, it has 16 reducible 3-vanishing characters of degree $3|G|_3$.

   (1) Ten characters of shape $\phi_1 = a\chi_7 + b\chi_8 + c\chi_9$, where $0 \leq a \leq b \leq c \leq 3$ and $a + b + c = 3$.
   (2) Six characters of shape $\phi_2 = \chi_1 + \chi_6 + d\chi_7 + e\chi_8 + f\chi_9$, where $0 \leq d \leq e \leq f \leq 2$ and $d + e + f = 2$.

This can be easily proved looking at the character table of G.

6. Syl_p-vanishing characters and Levi subgroups

Let $\mathbf{H}$ be a reductive group with Frobenius endomorphism Fr and $\mathbf{G}$ the maximal semisimple subgroup of $\mathbf{H}$. Set $H = \mathbf{H}^{Fr}$ and $G = \mathbf{G}^{Fr}$.

Lemma 6.1 ([18, Proposition p. 162]). *Let τ be an irreducible character of H. Then $\tau|_G$ is multiplicity free.*

Proposition 6.2. *Let G, H be as above. Let χ be a p-vanishing character of G. Then χ is H-invariant and extends to a Syl_p-vanishing character χ' of H such that $\chi'|_G = \chi$.*

Proof. Note that χ is an integral linear combination of the characters of projective indecomposable modules, and every of them extends to H (see [28, Lemma 3.14]). It follows that χ extends to a generalized p-vanishing character ϕ of H such that $\phi|_G = \chi$. This implies that χ is H-invariant. Next, we shall modify ϕ to obtain an ordinary Syl_p-vanishing character.

For $\sigma \in \operatorname{Irr} G$ denote by $\tilde{\sigma}$ the sum of the distinct H-conjugates of σ. As χ is H-invariant, we can write $\chi = \sum m(\tilde{\sigma})\tilde{\sigma}$ for some integers $m(\tilde{\sigma}) \geq 0$, with the sum over the minimal H-invariant characters of G.

Let $\tau \in \operatorname{Irr} H$. By Clifford's theorem, $\tau|_G = d_\tau \tilde{\sigma}$ for some integer $d_\tau > 0$. By Lemma 6.1, $d_\tau = 1$. Set $\operatorname{Irr}_{\tilde{\sigma}} H = \{\tau \in \operatorname{Irr} H : \tau|_G = \tilde{\sigma}\}$. Note that if $\tau_1, \tau_2 \in \operatorname{Irr}_{\tilde{\sigma}} H$ then $\tau_1|_G = \tau_2|_G = \tilde{\sigma}$. Then we have

$$\phi = \sum_{\tilde{\sigma}} \sum_{\tau \in \operatorname{Irr}_{\tilde{\sigma}} H} m(\tau)\tau,$$

where the first sum runs over the minimal H-invariant characters of G. Therefore, the multiplicity of $\tilde{\sigma}$ in ϕ equals $m(\tilde{\sigma}) = \sum_{\tau \in \operatorname{Irr}_{\tilde{\sigma}} H} m(\tau)$. Let $a_{\tilde{\sigma}}$ (resp., $b_{\tilde{\sigma}}$) be the sum of all positive (resp., negative) coefficients $m(\tau)$ occurring in this sum. Then $a_{\tilde{\sigma}} \geq b_{\tilde{\sigma}}$ as $m(\tilde{\sigma}) \geq 0$. Therefore, $\sum_{\tau \in \operatorname{Irr}_{\tilde{\sigma}} H} m(\tau)\tau|_G = (a_{\tilde{\sigma}} - b_{\tilde{\sigma}})\tau'|_G$ for any $\tau' \in \operatorname{Irr}_{\tilde{\sigma}} H$, and $(a_{\tilde{\sigma}} - b_{\tilde{\sigma}}) \cdot \tau'$ is an ordinary character of H. So we replace

every generalized character $\sum_{\tau \in \mathrm{Irr}_{\tilde{\sigma}} H} m(\tau)\tau$ by $(a_{\tilde{\sigma}} - b_{\tilde{\sigma}}) \cdot \tau'$ to obtain a required modification χ' of ϕ. $\qquad\square$

Remark. Proposition 6.2 is not true for Syl_p-vanishing characters. Indeed, the characters of $G = SL(2,q)$ in the item (3) of Lemma 5.2 do not extend to $GL(2,q)$.

Note that H is a group with split BN-pair, see [19, 24.10] or [3, p. 50]. So the Steinberg character can be defined for H as in [7, 66.35 and 71.2]. This is of degree $|H|_p$, and hence of defect 0.

Lemma 6.3. *Let χ be a Syl_p-vanishing character of H. Let L be a Levi subgroup of some parabolic subgroup P of H.*

    (1) *If $\chi \in \mathrm{Irr}\, H$ then $\chi(1) = |H|_p$, and $\chi|_G$ is the Steinberg character of G.*
    (2) *Suppose that the truncation $\overline{\chi}_L$ is irreducible. Then $\chi(1) = |H|_p$.*
    (3) *If $\chi \in \mathrm{Irr}\, H$ is of defect 0 then the truncation $\overline{\chi}_L$ is irreducible of defect 0.*

Proof. (1) is well known. (Indeed, $\chi(1)$ is divisible by $|H|_p$, so χ is of defect 0, and hence $\chi \pmod{p}$ is irreducible. The p-modular irreducible characters of H remain irreducible under restriction to G (cf. [28, 3.14]), and hence have degree at most $|G|_p = |H|_p$ (see [7, 72.19] or [25, Corollary of Theorem 46]). This implies (1).)

(2) As $\overline{\chi}_L$ is irreducible, it is of degree $|L|_p$ by (1). As $|H : P|$ is coprime to p, the claim follows from Lemma 2.5.

(3) As χ is of defect 0, the reduction of χ modulo p is irreducible and the corresponding module M is projective. By Smith's theorem [7, 72.32], the Harish-Chandra restriction $\overline{M}_L$ of M to L is irreducible and, moreover, projective. It is known that $\overline{\chi}_L$ is the character of $\overline{M}_L$, cf. [28, Lemma 3.3]. It follows that $\overline{\chi}_L$ is irreducible of defect 0. $\qquad\square$

Lemma 6.4. *Let ϕ be a defect 0 irreducible character of H, and L a Levi subgroup of H. Then $\phi = St$ if and only if $\overline{\phi}_L = St_L$.*

Proof. The "only if" part is well known, see for instance [9, p. 72]. For the "if" part suppose that $\overline{\phi}_L = St_L$. First we prove the lemma in the case where $L = T \subset B$.

Recall that $\phi(1) = |H|_p$ [7, 72.20], equivalently, $\overline{\phi}_T(1) = 1$ (Lemma 2.5). Suppose that $(\phi, 1_B) = 1$. By Lemma 6.3, $St_H|_G = St_G$ and $\phi|_G = St_G$. For every $h \in H$ we have $\phi(h) = St_H(h) \cdot \mu(h)$ for some complex number $\mu(h)$. It follows that the function μ is multiplicative (view ϕ and St_H as representations rather than characters). As H is a finite group, μ is a representation. Therefore, $\phi = St_H \cdot \mu$. As $U \subset G$, we have $\mu(U) = 1$. (Usually U is in the kernel of every linear character of H but there are a few exceptions.) Let $\phi = St \cdot \mu$. Then $1 = (\phi|_B, 1_B) = (St|_B \cdot \mu|_B, 1_B) = (St|_B, \mu^*|_B \cdot 1_B) = (St|_B, \mu^*|_B)$, where μ^* is the complex conjugate to μ. By the above, $\mu^*|_B = 1_B$. We show that $\mu^* = 1_H$.

Let H_0 be the subgroup generated by the unipotent elements in H. Then H/H_0 is a p'-group. As H_0B contains B, this is a parabolic subgroup of H, and $H_0B = H$ as every parabolic subgroup coincides with its normalizer [7, 65.19]. It follows that the mapping $\mu \to \mu|_B$ yields an injective mapping from $\mathrm{Irr}\, H/H_0$ to $\mathrm{Irr}\, B/B'$, where B' is a derived subgroup of B. Therefore, $\mu|_B = 1_B$ implies $\mu = 1_H$.

Let $L \neq T$ and let $\eta = \overline{\phi}_L$. By transitivity of truncation, $\overline{\phi}_T = \overline{\eta}_T$. By the above, $\phi = St$ is equivalent to $\overline{\phi}_T = 1_T$, and $\overline{\eta}_T = 1_T$ is equivalent to $\eta = St_L$. So the result follows. $\qquad\square$

We keep the notation H, G as in the beginning of the section, and assume that $\mathbf{G}$ is simply connected. In what follows we normally view H and G as groups with a split BN-pair defined by a Borel subgroup B and a subgroup N satisfying certain conditions, see [7, §65]. We set $T = N \cap B$ and $W = N/T$ (which is called the Weyl group). We denote by ℓ the BN-pair rank of H. (This is not the same as the rank of $\mathbf{H}$.) Set $I_\ell = \{1, \ldots, \ell\}$. The standard parabolic subgroups of H are those containing B. They form a lattice isomorphic to the lattice of the subsets of I_ℓ. We denote by P_J the standard parabolic subgroup corresponding to a subset J of I_ℓ (and B corresponds to the empty set). Let $U_J := O_p(P_J)$ be the unipotent radical of P_J. Then $P_J = U_J \rtimes L_J$ for some subgroup L_J called the Levi subgroup of P_J. The Levi subgroups are conjugate in P_J, and it is possible to choose L_J to contain T. (This is always assumed below.) If $K \subseteq J$, then $P_K \cap L_J$ is a standard parabolic subgroup of L_J containing the Borel subgroup $B \cap L_J$ of L_J. Furthermore, $O_p(P_K \cap L_J) = U_K \cap L_J$, $P_K \cap L_J = (U_K \cap L_J) \rtimes L_K$ and $U_K = U_J(U_K \cap L_J)$. In turn, L_J is a group with BN-pair whose Weyl group W_J is a subgroup of W. The Weyl group W has a unique (up to equivalence) irreducible representation ρ of degree ℓ called the reflection representation, and $\varepsilon := \det \circ \rho$ is called the sign character of W.

Let $P_J = L_J U_J$, where L_J is a Levi subgroup of P_J. Denote by G_J the normal subgroup of L_J generated by the unipotent elements of L_J. Then $L_J = G_J T$. Set $B_J = B \cap G_J$ and $T_J = T_0 \cap G_J$. Then B_J is a Borel subgroup of G_J and $B_J = (U \cap L_J) \rtimes T_J$. (Note that B_J differs from $B \cap L_J$, which is a Borel subgroup of L_J.)

Recall that we call G a Chevalley group if $\mathbf{G}$ is simple and simply connected.

In the previous section we obtained the following result for groups of BN-pair rank 1:

Proposition 6.5. *Let G be of BN-pair rank* 1. *Then G admits a reducible p-vanishing character χ of degree $|G|_p$, unless $G = SU(3,q)$ with $(q+1,3) = 1$. Furthermore, $(\chi, 1_G) = 1$, and some cuspidal character of G occurs as a constituent of χ.*

Proof. The result follows from Lemmas 5.2, 5.6, 5.7, 5.8 and 5.9. $\square$

In order to deal with groups of higher rank, we establish some properties of Syl_p-vanishing characters.

Lemma 6.6. *Suppose that $G = H$. Let $I_\ell = J \cup J'$ be the union of two subsets J, J'. Then $T = T_J T_{J'}$.*

Proof. The result follows from [14, Theorem 2.4.7(a)]. $\square$

Lemma 6.7. *Let χ be an arbitrary character of H, and let β_J be the truncation of $\overline{\chi}_{L_J}$ to T (so β_J is a character of T).*

(1) *$\beta_J = \overline{\chi}_T$.*
(2) *Set $\lambda = \overline{\chi}_{L_J}|_{G_J}$. Then $\overline{\lambda}_{T_J} = \beta_J|_{T_J}$.*

Proof. (1) follows from the transitivity of truncation.

(2) By the definition of truncation, for all $t \in T$ and $g \in G_J$ we have

$$\beta_J(t) = \frac{1}{|U|} \sum_{u \in U} \chi(ut) \ \text{ and } \ \lambda(g) = \frac{1}{|U_J|} \sum_{v \in U_J} \chi(vg).$$

So, for all $x \in T_J$,

$$(\overline{\lambda}_{T_J})(x) = \frac{1}{|U \cap L_J|} \sum_{y \in U \cap L_J} \lambda(yx) =$$

$$= \frac{1}{|U \cap L_J| \cdot |U_J|} \sum_{y \in U \cap L_J} \sum_{v \in U_J} \chi(vyx) = \frac{1}{|U|} \sum_{u \in U} \chi(ux) = \beta_J(x),$$

using the fact that $U = U_J(U \cap L_J)$. $\qquad\square$

Lemma 6.8. *Assume $G = H$. Let $I_\ell = J \cup J'$ be the union of two subsets J, J'. Let χ be a Syl_p-vanishing character of G. Suppose that $\overline{\chi}_{L_J}$ and $\overline{\chi}_{L_{J'}}$ are irreducible. Then $\overline{\chi}_T = 1_T$.*

Proof. Let $\eta = (\overline{\chi}_{L_J})|_{G_J}$ and $\eta' = (\overline{\chi}_{L_{J'}})|_{G_{J'}}$. Then $\eta = St_{G_J}$ and $\eta' = St_{G_{J'}}$ (Lemma 6.3(1) and (2)). By Lemma 6.4, the truncation of η to T_J is 1_{T_J} and the truncation of η' to $T_{J'}$ is $1_{T_{J'}}$. By Lemma 6.7, $\overline{\chi}_T|_{T_J} = 1_{T_J}$ and $\overline{\chi}_T|_{T_{J'}} = 1_{T_{J'}}$. Now Lemma 6.6 yields the result. $\qquad\square$

The following result is our primary target; this allows us to use classical results (quoted in Lemmas 6.12 and 6.13) on decomposition of 1_B^G as a sum of irreducible characters of G.

Proposition 6.9. *Let $G = H$ be and χ be a p-vanishing character of G of degree $|G|_p$. Then $(\chi, 1_B^G) = (1_B, \chi|_B) = 1$ (equivalently, $\overline{\chi}_T = 1_T$).*

Proof. The first equality is the Frobenius reciprocity. The second equality is equivalent to saying that $\overline{\chi}_T = 1_T$. We use induction on the BN-pair rank of G. By Proposition 6.5, the proposition is true for groups of rank 1. Let $I_\ell = J \cup J'$ be the union of two subsets J, J'. Then $\eta := \overline{\chi}_{L_J}|_{G_J}$ (resp., $\eta' := \overline{\chi}_{L_{J'}}|_{G_{J'}}$) is a p-vanishing character of G_J (resp., $G_{J'}$) of degree $|G_J|_p$ (resp., $|G_{J'}|_p$), see Lemma 2.5. By induction, the proposition is true for G_J and $G_{J'}$, so $\overline{\eta}_{T_J} = 1_{T_J}$ and $\overline{\eta'}_{T_{J'}} = 1_{T_{J'}}$. By Lemma 6.7, $\overline{\eta}_{T_J}$ coincides with $\overline{\chi}_T|_{T_J}$ (as L contains T). Therefore, $\overline{\chi}_T|_{T_J} = 1_{T_J}$ and $\overline{\chi}_T|_{T_{J'}} = 1_{T_{J'}}$. Now the result follows from Lemma 6.6. $\qquad\square$

Corollary 6.10. *Assume $G = H$. Let χ be a Syl_p-vanishing character of G such that $\overline{\chi}_T = 1_T$, and L a Levi subgroup of G.*

(1) *If $\overline{\chi}_L$ is irreducible of defect 0, then $\overline{\chi}_L = St_L$.*
(2) *If L is not solvable and $\overline{\chi}_L$ has a one-dimensional constituent ν, then $\nu = 1_L$.*

Proof. (1) By transitivity of Harish-Chandra restriction, the truncation of $\overline{\chi}_L$ to T coincides with 1_T. So the claim follows from Lemma 6.4 applied to L in place of G and T in place of L.

(2) Since the Borel subgroup of L is $(U \cap L)T$, by the definition of truncation, $\overline{\nu}_T = (\nu|_{U \cap L}, 1_{U \cap L}) \cdot \nu|_T$. Hence, $\nu|_T = 1_T$. On the other hand, let L_u be the subgroup of L generated by the unipotent elements of L. Then, one easily observes that $L = L_u \cdot T$. Since ν is a linear character of L, the derived subgroup of L coincides with L_u. It follows that $\nu|_{L_u} = 1_{L_u}$, so the result follows. $\qquad\square$

It is known that the blocks of non-zero defect of a finite reductive group H are in bijection with the elements of $Z(H)$, see [16, §8.7]; this is deduced from the fact that every such block has defect $|U|$ and $C_H(U) = Z(H)$. The following result shows that if a p-vanishing character of degree $|G|_p$ is in the principal p-block of G,

then the Harish-Chandra reduction $\overline{\chi}_L$ of χ to every Levi subgroup L belongs to the principal p-block of L.

Observe that Proposition 6.9 is a special case (for $L = T$) of the following result.

Corollary 6.11. *Let $G = H$ and L a Levi subgroup of G. Let χ be a p-vanishing character of G of degree $|G|_p$. Then $Z(L)$ is in the kernel of $\overline{\chi}_L$.*

Proof. As $Z(L)$ is contained in a Borel subgroup B_L of L, it is contained in a maximal torus T_L of B_L. However, T_L coincides with a maximal torus T of a Borel subgroup B of G. So $Z(L) \subset T$. By Proposition 6.9, $1_{Z(L)}$ is a constituent of $\overline{\chi}_L|_{Z(L)}$. Therefore, there is an irreducible constituent τ, say, of $\overline{\chi}_L$ such that $\tau|_{Z(L)} = \tau(1) \cdot 1_{Z(L)}$. In other words, $Z(L)$ is in the kernel of τ.

Suppose that the lemma is false. Then $Z(L)$ is not in the kernel of some irreducible constituent τ', say, of $\overline{\chi}_L$. Therefore, τ' and τ belong to distinct p-blocks. As $Z(L) \subset T$, it follows that $(|Z(L)|, p) = 1$. This is not the case by Lemma 2.1, as $\overline{\chi}_L$ is p-vanishing. $\qquad\square$

The following lemma is one of standard results on representations of groups with BN-pair, see [7, 68.24, pp. 683 - 684]. Recall that the Weyl group W of a group H with BN-pair has a natural representation as a linear group generated by reflections [7, 64.29]. Assigning to every $w \in W$ the determinant $\det w$ in this representation yields a one-dimensional representation of W called the sign representation. In the statement below the character of the sign representation of W is denoted by ε. For a subgroup X of W set $\varepsilon_X = \varepsilon|_X$.

Lemma 6.12. *Let H be a finite reductive group viewed as a group with BN-pair of rank ℓ, B a Borel subgroup of H and W the Weyl group of the BN-pair.*

(1) *There exists a bijection $\lambda \mapsto \chi_\lambda$ between $\operatorname{Irr} W$ and the set $\{\chi \in \operatorname{Irr} H : (\chi, 1_B^H) > 0\}$ such that $\chi_{1_W} = 1_H$ and $\chi_\varepsilon = St$. In addition, $(\chi_\lambda, 1_B^H) = \lambda(1)$.*

(2) *For each subset J of $I_\ell = \{1, \ldots, \ell\}$, let W_J and P_J be the corresponding subgroups of W and H, respectively. Then*

$$(\chi_\lambda, 1_{P_J}^H) = (\lambda, 1_{W_J}^W) = (\lambda|_{W_J}, 1_{W_J}).$$

In particular, $1_B^H = \sum_{\lambda \in \operatorname{Irr} W} \lambda(1) \cdot \chi_\lambda$.

(3) *[7, 70.24] $(\chi_\lambda, St_{L_J}^{\#H}) = (\lambda, \varepsilon_{W_J}^W) = (\lambda|_{W_J}, \varepsilon_{W_J})$.*

Lemma 6.13 ([7, (68.26)]). *Let W be in Lemma 6.12; assume that W is not a direct product of proper subgroups. Then each irreducible character λ of $\operatorname{Irr} W$ is uniquely determined by the multiplicities $\{(\lambda, 1_{W_J}^W) : J \subseteq I_\ell\}$, with the following exceptions:*

(1) *the characters of degree 2 if W is the dihedral group of order 12 or 16;*

(2) *the two characters of degree 2^9 if W is of type E_7;*

(3) *the four characters of degree 2^{12} if W is of type E_8.*

Corollary 6.14. *Keep notation of Lemma 6.12 and assumption of Lemma 6.13.*

(1) *Each irreducible character λ of $\operatorname{Irr} W$ is uniquely determined by the multiplicities $\{(\lambda, \varepsilon_{W_J}^W) : J \subseteq I_\ell\}$, with the exceptions $(1), (2), (3)$ in Lemma 6.13.*

(2) *The character χ_λ is uniquely determined by the multiplicities $\{(\chi_\lambda, 1^H_{P_J}) : J \subseteq I_\ell\}$ as well as the multiplicities $\{(\chi_\lambda, St^{\#H}_{L_J}) : J \subseteq I_\ell\}$, unless λ is listed in items $(1), (2), (3)$ of Lemma 6.13.*

Proof. (1) We have $(\lambda, \varepsilon^W_{W_J}) = (\lambda, (\varepsilon|_{W_J})^W) = (\lambda, 1^W_{W_J} \cdot \varepsilon) = (\lambda \cdot \varepsilon, 1^W_{W_J})$. By Lemma 6.13, $\lambda \cdot \varepsilon$ is determined by the multiplicities $\{(\lambda \cdot \varepsilon, 1^W_{W_J}) : J \subseteq I_\ell\}$, so λ is determined by $\{(\lambda, \varepsilon^W_{W_J}) : J \subseteq I_\ell\}$.

(2) follows from Lemmas 6.12 and 6.13, and statement (1) of this corollary. $\square$

The following lemma is a next important step in our analysis, as this settles a crucial special case, by showing that there is a Levi subgroup of G such that the Harish-Chandra restriction of χ to L is not irreducible.

Proposition 6.15. *Assume $G = H$. Let $I_\ell = J \cup K$ be the union of two subsets J, K. Let χ be a Syl_p-vanishing character of G.*

(1) *Suppose that $\overline{\chi}_{L_J}$ and $\overline{\chi}_{L_K}$ are irreducible. Then $\chi = St$.*
(2) *Suppose that $\overline{\chi}_T$ is irreducible and $(\overline{\chi}_{L_J}, 1_{L_J}) = (\overline{\chi}_{L_K}, 1_{L_K}) = 1$. Then $(\chi, 1_G) = 1$.*

Proof. (1) As $\overline{\chi}_{L_J}$ and $\overline{\chi}_{L_K}$ are irreducible and p-vanishing, they are of defect 0. By Lemma 6.3(2), $\overline{\chi}_{L_J}(1) = |L_J|_p$, and hence $\chi(1) = |G|_p$ by Lemma 2.5. Now, $\overline{\chi}_T = 1_T$ by Lemma 6.8, and $\overline{\chi}_{L_J} = St_{L_J}$ by Lemma 6.4. Similarly, $\overline{\chi}_{L_K} = St_{L_K}$. The equality $\overline{\chi}_T = 1_T$ also means that $(1^G_B, \chi) = 1$. Let $\tau \in \mathrm{Irr}\, G$ occur both in χ and 1^G_B (so $(\tau, 1^G_B) = 1$).

By Harish-Chandra reciprocity, $(\overline{\chi}_{L_J}, St_{L_J}) = (\chi, St^{\#G}_{L_J}) = 1$ and $(\overline{\chi}_{L_K}, St_{L_K}) = (\chi, St^{\#G}_{L_K}) = 1$. So τ occurs with multiplicity 1 in 1^G_B, $St^{\#G}_{L_J}$ and $St^{\#G}_{L_K}$. Let τ correspond to the linear character $\lambda \in \mathrm{Irr}\, W$ according to Lemma 6.12. By Lemma 6.12(3), $(\tau, St^{\#G}_{L_J}) = (\lambda|_{W_J}, \varepsilon_{W_J})$. Similarly, $(\tau, St^{\#G}_{L_K}) = (\lambda|_{W_K}, \varepsilon_{W_K})$. This is only possible if $\lambda = \varepsilon$, and hence $\tau = St$.

(2) Since $\overline{\chi}_T$ is irreducible and $(\overline{\chi}_{L_J}, 1_{L_J}) = 1$, we have $\overline{\chi}_T = 1_T$. We proceed as in item (1). Let $\tau \in \mathrm{Irr}\, G$ occur both in χ and 1^G_B (so τ corresponds to the linear character $\lambda \in \mathrm{Irr}\, W$ according to Lemma 6.12). By Lemma 6.12(2), $1 = (\tau, 1^G_{P_J}) = (\lambda|_{W_J}, 1_{W_J})$. Similarly, $(\lambda|_{W_K}, 1_{W_K}) = 1$. This is only possible if $\lambda = 1_W$, and hence $(\chi, 1_G) = 1$. $\square$

7. Chevalley groups of BN-pair rank 2

In this section, we deal with Chevalley groups of BN-pair rank 2. These are $SL(3, q)$, $Sp(4, q)$, $SU(4, q)$ $SU(5, q)$, $G_2(q)$, $^3D_4(q)$ and $^2F_4(q^2)$.

In view of Proposition 6.9 we state the following:

Lemma 7.1. *Let G be of BN-pair rank 2 and let ν be a multiplicity 1 irreducible constituent of 1^G_B. Then, either $\nu = 1_G$, or $\nu = St$ or of degree written below:*

$G = {}^2F_4(q^2)$: $\frac{q^2(q^6+1)(q^{12}+1)}{(q^4+1)(q^2+1)}$ *and* $\frac{q^{10}(q^6+1)(q^{12}+1)}{(q^4+1)(q^2+1)}$;

$G = {}^3D_4(q)$: $q^7(q^4 - q^2 + 1)$ *and* $q(q^4 - q^2 + 1)$;

$G = G_2(q)$: 2 *characters of degree* $q(q^4 + q^2 + 1)/3$;

$G = Sp(4, q)$: 2 *characters of degree* $q(q^2 + 1)/2$;

$G = SU(4, q)$: $q(q^2 - q + 1)$ *and* $q^3(q^2 - q + 1)$;

$G = SU(5,q)$: $q^2(q^5+1)(q+1)$ and $q^4(q^5+1)(q+1)$.

Proof. These degrees are given in [5, pp. 114 - 115]. $\qquad\square$

7.1. Groups $SL(3,q)$ and $GL(3,q)$. Note that $SL(3,q)$ is the only quasi-simple group of rank 2 where we can avoid substantial computations.

Proposition 7.2. *Let $H = GL(3,q)$ and let χ be a reducible Syl_p-vanishing character of H of degree q^3. Then $\chi = \mu + \sigma^{\#H}$, where μ is a linear character of H and σ is a cuspidal irreducible character of a proper Levi subgroup $L \neq T$ (and hence $\sigma(1) = q - 1$). In addition, $\sigma^{\#H}$ is irreducible.*

Let σ be a cuspidal character of L. Then the character $1_H + \sigma^{\#H}$ is p-vanishing if and only if $Z(L)$ is the kernel of σ.

Proof. Let L be a Levi subgroup of a parabolic subgroup P of H, where P is not a Borel subgroup. Then $L = XZ(L)$, where $X \cong GL(2,q)$.

By Proposition 6.15, $\overline{\chi}_L = \mu|_L + \sigma$, where $\mu(1) = 1$ and σ is cuspidal. (Indeed, as $L/Z(L) \cong PGL(2,q)$, it follows that L has no irreducible character of degree d with $1 < d \leq (q+1)/2$ for $q \neq 3$. In the latter case $(q-1)/2$ equals 1.)

By Harish-Chandra reciprocity, $(\chi, \sigma^{\#H}) = 1$. In addition, as $C_W(\sigma) = 1$, the Harish-Chandra series of σ consists of a single character, see [7, p. 678]. (Note that in our case the ramification group defined in [7, p. 678] is trivial as $N_H(L) = L$.) Therefore, $\sigma^{\#H}$ is irreducible, and $\sigma^{\#H}(1) = q^3 - 1$. As $\chi(1) = q^3$, the first statement of the lemma follows.

Let σ be any cuspidal character of L. It is easy to compute that $\sigma^{\#H}(u) = -1$ for every unipotent element $u \neq 1$. So $\mu + \sigma^{\#H}$ is Syl_p-vanishing for every linear character μ of H.

Suppose that $\chi = 1_H + \sigma^{\#H}$ is p-vanishing. Then, by Corollary 6.11, σ is trivial on $Z(L)$. Conversely, let $g = su \in H$, where $s \neq 1$ is semisimple, u is unipotent and $su = us$. Then g is conjugate to an element of L, and if $g \in L$ then $s \in Z(L)$. Therefore, $\sigma(su) = \sigma(u) = -1$. Let C be the conjugacy class of su in H. One observes that $C \cap P$ forms a conjugacy class in P. Let τ be the inflation of σ to P. Then $\tau^H = \sigma^{\#H}$ and $\tau(su) = -1$. By the induced character formula, we have $\tau^H(su) = -1$, which proves the second statement. $\qquad\square$

Corollary 7.3. *Let $G = SL(3,q)$, and let χ be a p-vanishing character of degree q^3. Then either $\chi = St$ or $(\chi, 1_G) = 1$. In addition, no irreducible constituent of χ is cuspidal.*

Proposition 7.2 can be compared with the results obtained with use of CHEVIE. Below are some details.

Let $H = GL(3,q)$. Following CHEVIE, we partition Irr H into 8 sets $\mathbf{X}_h$, $1 \leq h \leq 8$, see Table 11.5.1-A. The Syl_p-decompositions of Table 11.5.1-B allow us to work only with the characters χ_h for $h \in \{1,2,7,8\}$. For this range of h, it is immediate to observe that $\psi_1 = \chi_1 + \chi_7$ is the unique Syl_p-vanishing character of degree $|H|_p$. Now, applying the Syl_p-decompositions from Table 11.5.1-B to this character, we obtain all Syl_p-vanishing characters of H of degree $|H|_p$ (see Table 11.5.1-C). In particular, every such character has a one-dimensional irreducible constituent.

Lemma 7.4. *Let $H = GL(3,q)$. Then H admits exactly $(q-1)(q-\delta)/2$ reducible p-vanishing characters ψ of degree $|H|_p$, all of type*

$$\psi = \chi_1(n_1) \cdot (1_H + \chi_7(q-1, n_7)),$$

where $\delta = \frac{1-(-1)^q}{2}$, $n_1 \in \mathbb{Z}_{q-1}$ *and* $n_7 = (q-1)n_7'$, *with* $n_7' \in \mathbb{Z}_{q+1}$ *and* $q+1$ *does not divide* n_7'.

Proof. Modulo linear characters of H, take $\psi_1 = 1_H + \chi_7(m,n)$. Then, $(q-1) \mid (m+n)$ and it suffices to look at the classes $C_5(a,b)$, where we have $\zeta_1^{m(b-a)} = 1$. Thus, $m = q-1$ and $n = (q-1)n'$, where $n' \in \mathbb{Z}_{q+1}$ and $q+1$ does not divide n'. $\qquad\square$

Now, let $G = SL(3,q)$. We have to distinguish two cases, depending on whether 3 divides $q-1$ or not. However, the computations remain very similar to those for $GL(3,q)$. We provide the relevant informations in Tables 11.5.2-A,B,C if $3 \mid (q-1)$, and in Tables 11.5.3-A,B,C otherwise. Thus, it can be easily proved the following

Lemma 7.5. *Let* $G = SL(3,q)$ *and* $\delta = \frac{1-(-1)^q}{2}$. *Then,* G *admits exactly* $(q-\delta)/2$ *reducible p-vanishing characters* ψ *of degree* $|G|_p$, *all of type*

$$\psi = 1_G + \chi_i(a),$$

where $a = (q+1)a'$, $i = 10$ *if* $3 \mid (q-1)$, *and* $i = 7$, *otherwise.*

7.2. Groups $H = U(4,q)$.

CHEVIE partitions $\mathrm{Irr}\,H$ into 23 sets $\mathbf{X}_h$, see Table 11.6-A. Let $\psi = \sum \chi_h$ be a reducible Syl_p-vanishing character of H of degree $|H|_p$.

Due to the Syl_p-decompositions in Table 11.6-B, we first assume that $h \in \{2, 5, 7, 8, 12, 19, 23\}$. Note that χ_{19} and χ_{23} are the only regular characters of this set. However, since the degree of χ_{23} is greater than $|H|_p$, it follows that χ_{19} is a constituent of ψ. So the other constituents of ψ are $\{\chi_h \mid h = 2, 5, 7, 8, 12\}$. Using the classes $C_3(q+1)$ and $C_5(q+1)$, we observe that ψ must have at least 2 constituents of type χ_7 and a constituent of type χ_8. Now, the character values at the class $C_4(q+1)$ imply that either χ_2 is a constituent of ψ, or both χ_8 and χ_{12} are other constituents of ψ. In the latter case we get a contradiction on the class $C_3(q+1)$. Therefore, we get a unique Syl_p-vanishing character of degree $|H|_p$: $\psi_1 = \chi_2 + \chi_7 + \chi_7 + \chi_8 + \chi_{19}$. Applying the Syl_p-decompositions of Table 11.6-B to ψ_1, we determine all Syl_p-vanishing characters of H of degree $|H|_p$ (see Table 11.6-C).

Lemma 7.6. *Let* $H = U(4,q)$. *Then neither* H *nor* $G = H'$ *has a reducible p-vanishing character of degree* $|H|_p$.

Proof. Suppose the contrary and assume $q \geq 4$ (the cases $q = 2, 3$ can be solved with MAGMA). In view of Table 11.6-C, up to linear characters of H, we have to consider the following four cases:

(1) $\psi_1 = \chi_2(0) + \chi_7(k_{7,1}, k_{7,2}) + \chi_7(k_{7,1}', k_{7,2}') + \chi_8(k_{8,1}, k_{8,2}) + \chi_{19}(k_{19,1}, k_{19,2}, k_{19,3}, k_{19,4})$. On the class $C_{18}(1,2)$ we obtain $\xi_1^{k_{8,1}-k_{8,2}} = 1$. This implies that $q+1$ divides $k_{8,1} - k_{8,2}$, a contradiction.

(2) $\psi_2 = \chi_2(0) + \chi_7(k_{7,1}, k_{7,2}) + \chi_{16}(k_{16,1}, k_{16,2}, k_{16,3}) + \chi_{19}(k_{19,1}, k_{19,2}, k_{19,3}, k_{19,4})$. On the classes $C_{16}(q+1, i_2, -i_2)$ we have

$$2 + Re(\xi_1^{(k_{7,1}-k_{7,2})i_2}) + Re(\xi_1^{(k_{16,1}-k_{16,2})i_2}) + Re(\xi_1^{(k_{16,1}-k_{16,3})i_2}) +$$

$$+ Re(\xi_1^{(k_{16,2}-k_{16,3})i_2}) = Re(\xi_1^{(k_{19,1}-k_{19,2})i_2}) + Re(\xi_1^{(k_{19,1}-k_{19,3})i_2}) +$$

$$+ Re(\xi_1^{(k_{19,1}-k_{19,4})i_2}) + + Re(\xi_1^{(k_{19,2}-k_{19,3})i_2}) + Re(\xi_1^{(k_{19,2}-k_{19,4})i_2}) +$$

$$+ Re(\xi_1^{(k_{19,3}-k_{19,4})i_2}).$$

      This holds for every $i_2 \in \mathbb{Z}_{q+1}$ such that $q+1$ does not divide $2i_1$. If $q \geq 13$, this implies that $q+1$ divides $k_{19,i} - k_{19,j}$ for some $i \neq j$, a contradiction. For $4 \leq q \leq 11$, it can be proved directly that this condition for $i_1 = 1, \ldots, \lfloor (q-1)/2 \rfloor$ is never satisfied.

(3) $\psi_3 = \chi_2(0) + \chi_7(k_{7,1}, k_{7,2}) + \chi_{15}(k_{15,1}, k_{15,2}, k_{15,3})$. On the class $C_8(1, q-2)$ we have $\xi_1^{k_{7,1}-k_{7,2}} = \xi_1^{2k_{15,1}-k_{15,2}-k_{15,3}}$ and on $C_{18}(1,2)$ we have $1 = \xi_1^{2k_{15,1}-k_{15,2}-k_{15,3}}$. So $\xi_1^{k_{7,1}-k_{7,2}} = 1$, and hence $(q+1) \mid (k_{7,1} - k_{7,2})$, a contradiction.

(4) $\psi_4 = \chi_2(0) + \chi_6(k_1, k_2)$. Using the class $C_7(1, q-2)$ we have $\xi_1^{k_2-k_1} = 1$. It follows that $(q+1) \mid (k_2 - k_1)$, a contradiction with the conditions on the parameters.

Since all the conjugacy classes we have considered belong to H', it follows from Proposition 6.2 that also H' has no p-vanishing character of degree $|H|_p$.   □

7.3. Groups $H = U(5,q)$.

CHEVIE partitions $\mathrm{Irr}\, H$ into 43 sets, see Table 11.7-A. Let $\psi = \sum \chi_h$ be a reducible Syl_p-vanishing character of H of degree $|H|_p$.

    The Syl_p-decompositions of Table 11.7-B allow us to deal only with the characters χ_h for $h \in \{2, 5, 6, 7, 16, 20, 21, 37, 42, 43\}$. Note that χ_{37}, χ_{42} and χ_{43} are the only regular characters in this set. However, χ_{43} is ruled out by degree reasons.

(1) Suppose that χ_{37} is a constituent of ψ. Then, looking at the classes $C_3(q+1)$ and $C_7(q+1)$, the character ψ must have at least one constituent of type χ_{21} and three constituents of type χ_{20}. With some little efforts, we get a unique Syl_p-vanishing character $\psi_1 = \chi_2 + \chi_5 + \chi_{16} + \chi_{20} + \chi'_{20} + \chi''_{20} + \chi_{21} + \chi_{37}$, where $\chi_{20}, \chi'_{20}, \chi''_{20} \in \mathbf{X}_{20}$ are not necessarily distinct.

(2) Let χ_{42} be a constituent of ψ. Then, by degree reasons, the other constituents of ψ must belong to the set $\{\chi_h \mid h = 5, 6, 7, 16, 21\}$. Then, looking at the classes $C_5(q+1)$ and $C_7(q+1)$, the character ψ must have one constituent of type χ_7 and at least one constituent of type χ_{16}. This leads to a contradiction on the class $C_6(q+1)$.

Now, applying the Syl_p-decompositions of Table 11.7-B to the character ψ_1, we obtain all Syl_p-vanishing characters of H of degree $|H|_p$ (see Table 11.7-C).

Lemma 7.7. *Let $H = U(5,q)$. Then, neither H nor $G = H'$ admit any reducible p-vanishing character of degree $|H|_p$.*

Proof. Suppose the contrary. The cases $q = 2, 3$ can be solved using MAGMA. Assume $q > 3$. First, take $\psi_{14} = \chi_2(k_{2,1}) + \chi_8(k_{8,1}, k_{8,2})$. Using the class $C_9(1, q-3)$ we have $\xi_1^{k_{8,2}-k_{8,1}} = 1$, a contradiction. In all the other cases, it suffices to consider the classes $C_{34}(q+1, i_2, i_2)$. We obtain one of the two following conditions:

(a) $1 + \xi_1^{(k_{a,2}-k_{a,1})i_2} = \xi_1^{(k_{b,1}-k_{b,2})i_2} + \xi_1^{(k_{b,1}-k_{b,3})i_2}$, $i_2 = 1, 2$;

(b) $2 + \xi_1^{(k_{a,2}-k_{a,1})i_2} = \xi_1^{(k_{b,1}-k_{b,2})i_2} + \xi_1^{(k_{b,1}-k_{b,3})i_2} + \xi_1^{(k_{b,1}-k_{b,4})i_2}$, $i_2 = 1, 2, 3$.

    In both the cases, this implies that $q+1$ divides $k_{b,1} - k_{b,j}$ for some $j \neq 1$, a contradiction with the conditions on the parameters. (The case (b) for $q = 4$ needs to be solved directly.)

Since all the conjugacy classes that we have considered belong to H', if follows from Proposition 6.2 that also $G = H'$ has no p-vanishing characters of degree $|G|_p$.   □

7.4. Groups $G = Sp(4, q)$. Let $G = Sp(4, q)$ with q even. According to CHEVIE, we partition $\operatorname{Irr} G$ into 19 sets, see Table 11.8.1-A. Let ψ be a reducible Syl_p-vanishing character of G of degree $|G|_p$. The Syl_p-decompositions of Table 11.8.1-B allow us to work only with the characters χ_h where $h \in \{1, 2, 3, 4, 5, 9, 13, 18, 19\}$. Note that the regular characters belonging to this set are χ_{18} and χ_{19}.

(1) Let χ_{18} be a constituent of ψ. Then, by degree reasons, the other constituents of ψ must belong to the set $\{\chi_h \mid h = 1, 5\}$. In this case we easily obtain a contradiction.

(2) Let χ_{19} be a constituent of ψ. Then, by degree reasons, the other constituents of ψ must belong to the set $\{\chi_h \mid h = 1, 2, 3, 4, 5, 9, 13\}$. Looking at the class C_5, either χ_9 or χ_{13} is a constituent of ψ. In the former case, we obtain the Syl_p-vanishing characters $\psi_1 = \chi_1 + \chi_9 + \chi_{13} + \chi_{19}$ and $\psi_2 = \chi_3 + \chi_5 + \chi_9 + \chi_{19}$. In the latter case, we obtain the characters ψ_1 and $\psi_3 = \chi_4 + \chi_5 + \chi_{13} + \chi_{19}$.

Now, applying the Syl_p-decompositions of Table 11.8.1-B to the characters ψ_1, ψ_2 and ψ_3, we are able to list all the Syl_p-vanishing characters of G of degree $|G|_p$ (see Table 11.8-C).

Lemma 7.8. *Let $G = Sp(4, q)$, q even. Then G has no reducible p-vanishing character of degree $|G|_p$.*

Proof. Suppose the contrary. Then we have to consider the following cases.

(1) $\psi_1 = 1_G + \chi_9(k_9) + \chi_{13}(k_{13}) + \chi_{19}(k_{19}, l_{19})$. For $i = 1, 2$, on the classes $C_{10}(i)$ we have $Re(\xi_1^{ik_{19}}) + Re(\xi_1^{il_{19}}) = Re(\xi_1^{ik_{13}}) + Re(\xi_1^{ik_9})$ and on the classes $C_{14}(i)$ we have $Re(\xi_1^{(k_{19}+l_{19})i}) + Re(\xi_1^{(k_{19}-l_{19})i}) = Re(\xi_1^{2ik_{13}}) + Re(\xi_1^{ik_9})$. Set $A = Re(\xi_1^{k_{19}})$, $B = Re(\xi_1^{l_{19}})$, $C = Re(\xi_1^{k_{13}})$ and $D = Re(\xi_1^{k_9})$:

$$\begin{cases} 2AB = 2C^2 - 1 + D \\ (2AB)^2 - 2(A^2 + B^2) + 1 = 4C^4 - 4C^2 + D^2 \\ (A, B) = (C, D), (D, C) \end{cases}.$$

If $C = \frac{1}{2}$, then $2AB = D - \frac{1}{2}$. This implies that either $A = C = \frac{1}{2}$ and $B = D = D - 1/2$ or $B = C = \frac{1}{2}$ and $A = D = D - \frac{1}{2}$. So, in both the cases we have an absurd. Hence, we may assume $C \neq \frac{1}{2}$ and $D = \frac{2C^2-1}{2C-1}$. We get the equation

$$8(C - 1)^2 \left(C^3 + \frac{1}{2}C^2 - \frac{1}{2}C - \frac{1}{8} \right) = 0.$$

Now, since $C \neq 1$:

(7.1) $\quad 0 = C^3 + \frac{1}{2}C^2 - \frac{1}{2}C - \frac{1}{8} = \left(C - \cos(\frac{2\pi}{7}) \right)\left(C - \cos(\frac{4\pi}{7}) \right)\left(C - \cos(\frac{6\pi}{7}) \right).$

Thus, we have a solution if and only if 7 divides $q + 1$. However, this does not happen since q is even.

(2) $\psi_2 = 1_G + \chi_9(k_9) + \chi_{14}(k_{14})$. On the class $C_{14}(1)$ we have $Re(\xi_1^{k_9}) = 1$, a contradiction.

(3) $\psi_3 = \chi_1 + \chi_{10}(k_{10}) + \chi_{13}(k_{13})$. On the class $C_{10}(1)$ we have $Re(\xi_1^{k_{13}}) = 1$, a contradiction.

(4) Take $\psi_4 = \chi_3 + \chi_5 + \chi_9(k_9) + \chi_{19}(k_{19}, l_{19})$. On the classes $C_{14}(i)$ $(i = 1, 2)$ we have $1 + Re(\xi_1^{2ik_9}) = Re(\xi_1^{i(k_{19}+l_{19})}) + Re(\xi_1^{i(k_{19}-l_{19})})$, which implies that $q + 1$ divides either $k_{19} + l_{19}$ or $k_{19} - l_{19}$. In both cases, we get a contradiction.

(5) $\psi_5 = \chi_3 + \chi_5 + \chi_{10}(k)$. On the class $C_{14}(1)$ we have $Re(\xi_1^k) = 1$, a contradiction.

(6) $\psi_6 = \chi_4 + \chi_5 + \chi_{13}(k_{13}) + \chi_{19}(k_{19}, l_{19})$. On the classes $C_{14}(i)$ $(i = 1, 2)$ we have $1 + Re(\xi_1^{2ik_{13}}) = Re(\xi_1^{i(k_{19}+l_{19})}) + Re(\xi_1^{i(k_{19}-l_{19})})$, which implies that $q + 1$ divides either $k_{19} + l_{19}$ or $k_{19} - l_{19}$. In both cases, we get a contradiction.

(7) $\psi_7 = \chi_4 + \chi_5 + \chi_{14}(k)$. On the class $C_{10}(1)$ we have $Re(\xi_1^k) = 1$, a contradiction.

$\square$

Now, let $G = Sp(4, q)$, q odd. The character table of this group is not implemented in CHEVIE, but it was described in papers by Srinivasan [24], Przygocki [22] and Bleher [2]. We keep the notation of [24] for the conjugacy classes.

Let ψ be a reducible Syl_p-vanishing character of degree $|G|_p$. We partition $\mathrm{Irr}\, G$ into 49 sets $\mathbf{X}_h$ as done in Table 11.8.2-A. (Note that this is not taken from CHEVIE but again all characters in every set are Syl_p-equivalent.) The Syl_p-decompositions of Table 11.8.2-B allow us to work only with the characters χ_h, where $h \in \{6, 16, 17, 37, 38, 40, 42, 44\}$. Note that the only regular characters in this set are χ_{16} and χ_{17}. By Lemma 3.1, at least one of them is a constituent of ψ. Inspecting [23], we observe that χ_{16} and χ_{17} are $CSp(4, q)$-conjugate. By Proposition 6.2, both of them are constituents of ψ. (It follows that they are constituents of two distinct non-degenerate Gelfand-Graev characters.) Working only with these constituents, we obtain only two Syl_p-vanishing characters of degree $|G|_p$:

$$\psi_1 = \chi_6 + \chi_{16} + \chi_{17} + \chi_{37} + \chi_{38} + \chi_{40} + \chi_{40} + \chi_{44}$$

and

$$\psi_2 = \chi_{16} + \chi_{17} + \chi_{37} + \chi_{38} + \chi_{40} + \chi_{40} + \chi_{40} + \chi_{42}.$$

Now, applying the Syl_p-decompositions of Table 11.8.2-B to the characters ψ_1 and ψ_2, we obtain all Syl_p-vanishing characters of G of degree $|G|_p$ (see Table 11.8.2-C).

Lemma 7.9. *Let $G = Sp(4, q)$, q odd. Then G admits reducible p-vanishing characters ψ of degree $|G|_p$ if and only if $7 \mid (q + 1)$. In this case, there are exactly 3 characters ψ, all of type*

$$\psi = 1_G + \chi_4(a, b) + \chi_6(c) + \chi_{10}(d),$$

where

$$(a, b, c, d) \in \{(T, 3T, T, 3T), (T, 2T, 2T, T), (2T, 3T, 3T, 2T)\}$$

and $T = (q + 1)/7$.

Proof. Most of the Syl_p-vanishing characters of G can be easily ruled out. We present here some examples for the most interesting cases.

(1) Let $\psi_1 = 1_G + \chi_4(k_4, l_4) + \chi_6(k_6) + \chi_{10}(k_{10})$. On the class D_{31} we have $(-1)^{k_4} + (-1)^{l_4} = (-1)^{k_6} + (-1)^{k_{10}}$ On the class D_{21} we have $(q-1)(-1)^{k_6} +$

$q = (q-1)((-1)^{k_4} + (-1)^{l_4}) + (-1)^{k_{10}}$. Thus, k_{10} is even. On the class $B_7(i)$ we have

$$Re(\xi_1^{(k_4+l_4)i}) + Re(\xi_1^{(k_4-l_4)i}) = Re(\xi_1^{2ik_6}) + Re(\xi_1^{ik_{10}})$$

and on the class $C_{21}(i)$:

$$Re(\xi_1^{k_4 i}) + Re(\xi_1^{l_4 i}) = Re(\xi_1^{ik_6}) + Re(\xi_1^{ik_{10}}).$$

As for the case q even, we set $A = Re(\xi_1^{k_4})$, $B = Re(\xi_1^{l_4})$, $C = Re(\xi_1^{k_6})$ and $D = Re(\xi_1^{k_{10}})$, obtaining $C = \cos(2t\pi/7)$, $t = 1, 2, 3$, and $D = \cos(6t\pi/7)$. Thus,

$$
\begin{aligned}
(A, B, C, D) \quad \in \quad & \Big\{ \Big(\cos \frac{2\pi}{7}, \cos \frac{6\pi}{7}, \cos \frac{2\pi}{7}, \cos \frac{6\pi}{7} \Big), \\
& \Big(\cos \frac{6\pi}{7}, \cos \frac{2\pi}{7}, \cos \frac{2\pi}{7}, \cos \frac{6\pi}{7} \Big), \\
& \Big(\cos \frac{4\pi}{7}, \cos \frac{2\pi}{7}, \cos \frac{4\pi}{7}, \cos \frac{2\pi}{7} \Big), \\
& \Big(\cos \frac{2\pi}{7}, \cos \frac{4\pi}{7}, \cos \frac{4\pi}{7}, \cos \frac{2\pi}{7} \Big), \\
& \Big(\cos \frac{6\pi}{7}, \cos \frac{4\pi}{7}, \cos \frac{6\pi}{7}, \cos \frac{4\pi}{7} \Big), \\
& \Big(\cos \frac{4\pi}{7}, \cos \frac{6\pi}{7}, \cos \frac{6\pi}{7}, \cos \frac{4\pi}{7} \Big) \Big\}.
\end{aligned}
$$

This means, denoting $T = \frac{q+1}{7}$, that

$$
\begin{aligned}
(k_4, l_4, k_6, k_{10}) \quad \in \quad & \{ (T, 3T, T, 3T), (3T, T, T, 3T), (2T, T, 2T, T) \\
& (T, 2T, 2T, T), (3T, 2T, 3T, 2T), (2T, 3T, 3T, 2T) \}.
\end{aligned}
$$

It is clear now that $Re(\xi_1^{(k_4+l_4)i}) + Re(\xi_1^{(k_4-l_4)i}) = Re(\xi_1^{2ik_6}) + Re(\xi_1^{ik_{10}})$ and $Re(\xi_1^{ik_4}) + Re(\xi_1^{il_4}) = Re(\xi_1^{ik_6}) + Re(\xi_1^{ik_{10}})$ for all i. It follows that ψ is a p-vanishing character if and only if 7 divides $q + 1$. In this case, since $\chi_4(a, b) = \chi_4(b, a)$, G admits exactly three p-vanishing characters of degree $|G|_p$ as described in the statement.

(2) Take $\psi_{16} = 1_G + \chi_6(k_6) + \chi_{10}(k_{10}) + \chi_{16}(k_{16}) + \chi_{17}(k_{17})$. Looking at classes $C_{21}(1)$ and $C_{22}(1)$ we obtain $Re(\xi_1^{k_{16}}) = Re(\xi_1^{k_{17}})$. On the classes $B_7(i)$ ($i = 1, 2$) we get $Re(\xi_1^{2ik_6}) + Re(\xi_1^{ik_{10}}) = 2(-1)^i Re(\xi_1^{ik_{16}})$. Hence $Re(\xi_1^{k_{10}}) = -1$, a contradiction.

(3) $\psi_{29} = 1_G + \chi_6(k_6) + \chi_{16}(k_{16}) + \chi_{23} + \chi_{24}$. On the classes $C_{21}(1)$ and $C_{22}(1)$ we obtain $Re(\xi_1^{k_{16}}) = 1$, a contradiction.

$\square$

Remark. It can be shown that the p-vanishing characters ψ_1 of $Sp(4, q)$, described above, can be extended to p-vanishing characters of $CSp(4, q)$. In Tables 11.8.3-A,B,C we report information about irreducible characters, Syl_p-decompositions and Syl_p-vanishing characters of $CSp(4, q)$, q odd.

7.5. **Groups $^3D_4(q)$.** Let $G = {}^3D_4(q)$. The character table of G depends on the parity of q. Here we describe in detail only the case q odd; the case q even is very similar (see Tables 11.9-1.A,B,C).

So, let q be odd. According to CHEVIE, we partition $\operatorname{Irr} G$ into 33 sets, as listed in Table 11.9.2-A. Let $\psi = \sum \chi_h$ be a reducible Syl_p-vanishing character of G of degree $|G|_p$. The Syl_p-decompositions of Table 11.9.2-B reduce our search to the case where $h \in \{1, 2, 3, 4, 5, 6, 7, 9, 15, 16, 21, 24, 25, 30, 33\}$, which we assume below. Note that, for these h, the only regular characters χ_h are with $h = 30, 33$.

First, some computation shows that $(\chi_{30}, \psi) = 0$.

Next, assume that χ_{33} is a constituent of ψ. Looking at the values of χ_h at the class C_7, we observe that χ_{21} or χ_{24} is a constituent of ψ. In the former case, we get the character $\psi_1 = \chi_1 + \chi_{21} + \chi_{24} + \chi_{25} + \chi_{33}$. In the latter case, we obtain the characters $\psi_2 = \chi_5 + \chi_6 + \chi_7 + \chi_{21} + \chi_{33}$ and $\psi_3 = \chi_2 + \chi_5 + \chi_6 + \chi_{24} + \chi_{25} + \chi_{33}$.

Finally, applying the Syl_p-decompositions of Table 11.9.2-B to the characters ψ_1, ψ_2, ψ_3, we determine all Syl_p-vanishing characters of G of degree $|G|_p$ (see Table 11.9.2-C).

Lemma 7.10. *The group $G = {}^3D_4(q)$ has no reducible p-vanishing character of degree $|G|_p$.*

Proof. Suppose the contrary. Assume that q is odd (the case of q even is similar).

(1) Take $\psi_1 = 1_G + \chi_{21}(k_{21}) + \chi_{24}(k_{24}) + \chi_{25}(k_{25}) + \chi_{33}(k_{33}, l_{33})$. On the classes $C_{22}(a)$ $(a = 1, 2, 3)$ we have

$$1 + Re(\xi_1^{ak_{21}}) + Re(\xi_1^{2ak_{21}}) = Re(\xi_1^{al_{33}}) + Re(\xi_1^{a(k_{33}+l_{33})}) + Re(\xi_1^{a(k_{33}+2l_{33})}).$$

This implies that $q + 1$ divides at least one of the numbers l_{33}, $k_{33} + l_{33}$, $k_{33} + 2l_{33}$, which violates the description of the parameter set I_h.

(2) Take $\psi_2 = \chi_5 + \chi_6 + \chi_7 + \chi_{21}(k_{21}) + \chi_{33}(k_{33}, l_{33})$. In this case, we have the same contradiction as in the previous case.

(3) Take $\psi_3 = \chi_2 + \chi_5 + \chi_6 + \chi_{24}(k_{24}) + \chi_{25}(k_{25}) + \chi_{33}(k_{33}, l_{33})$. On the class $C_{22}(1)$ we obtain $Re(\xi_1^{l_{33}}) + Re(\xi_1^{k_{33}+l_{33}}) + Re(\xi_1^{k_{33}+2l_{33}}) = 3$. This leads to a contradiction.

(4) Take $\psi_4 = 1_G + \chi_{21}(k_{21}) + \chi_{27}(k_{27}) + \chi_{33}(k_{33}, l_{33})$. On the class $C_{22}(a)$ we get

$$Re(\xi_1^{ak_{21}}) + Re(\xi_1^{2ak_{21}}) + Re(\xi_1^{ak_{27}}) = Re(\xi_1^{al_{33}}) + Re(\xi_1^{a(k_{33}+l_{33})}) + Re(\xi_1^{a(k_{33}+2l_{33})}).$$

Using the classes $C_{25}(1)$ and $C_{26}(1)$ we get $Re(\varphi_6^{k_{27}}) = Re(\varphi_6^{k_{33}}) = 1$. Furthermore, setting $k_{27} = (q^2 - q + 1)k'_{27}$ and $k_{33} = (q^2 - q + 1)k'_{33}$, on the classes $C_{28}(a)$ we have

$$Re(\xi_1^{ak_{21}}) + Re(\xi_1^{ak'_{27}}) + Re(\xi_1^{2ak'_{27}}) = Re(\xi_1^{ak'_{33}}) + Re(\xi_1^{a(k'_{33}+l_{33})}) + Re(\xi_1^{a(2k'_{33}+l_{33})}).$$

With some work, this yields a contradiction.

(5) Take $\psi_5 = \chi_1 + \chi_{21}(k_{21}) + \chi_{28}(k_{28})$. On the classes $C_{22}(a)$ $(a = 1, 2)$ we obtain $1 + Re(\xi_1^{ak_{28}}) = Re(\xi_1^{ak_{21}}) + Re(\xi_1^{2ak_{21}})$, which implies $(q+1)/2 \mid k_{21}$, a contradiction.

(6) Take $\psi_6 = 1_G + \chi_{22}(k_{22}) + \chi_{24}(k_{24}) + \chi_{25}(k_{25})$. On the class $C_{22}(1)$ we obtain the equality $Re(\xi_1^{k_{22}}) = 1$, which is a contradiction.

(7) Take $\psi_7 = 1_G + \chi_{22}(k_{22}) + \chi_{27}(k_{27})$. On the class $C_{22}(1)$ we obtain $Re(\xi_1^{k_{22}}) = Re(\xi_1^{k_{27}})$ and on the class $C_{26}(1)$ we get $Re(\varphi_6^{k_{27}}) = 1$. On the class $C_{28}(1)$ we have $Re(\xi_3^{q(q-1)k_{27}}) = 1$. This implies that $(q^3+1)/2$ divides k_{27}, which is a contradiction.

(8) Take $\psi_8 = \chi_2 + \chi_5 + \chi_6 + \chi_{27}(k_{27}) + \chi_{33}(k_{33})$. On the classes $C_{22}(a)$ $(a = 1, 2, 3)$ we get

$$2 + Re(\xi_1^{ak_{27}}) = Re(\xi_1^{al_{33}}) + Re(\xi_1^{a(k_{33}+l_{33})}) + Re(\xi_1^{a(k_{33}+2l_{33})}).$$

This implies that $q+1$ divides at least one of the numbers l_{33}, $k_{33} + l_{33}$, $k_{33} + 2l_{33}$, which is a contradiction.

(9) Take $\psi_9 = \chi_2 + \chi_5 + \chi_6 + \chi_{28}(k_{28})$. On the class $C_{22}(1)$ we have $Re(\xi_1^{k_{28}}) = 1$, a contradiction.

(10) Take $\psi_{10} = \chi_5 + \chi_6 + \chi_7 + \chi_{22}(k_{22})$. On the class $C_{22}(1)$ we obtain $Re(\xi_1^{k_{22}}) = 1$, a contradiction.

$\square$

7.6. Groups $G_2(q)$.

Let $G = G_2(q)$. The character table of G provided by CHEVIE distinguishes five cases, depending on the parity of q and on the congruence class of q modulo 3. All the cases are very similar, so we choose to describe in detail only the case q odd with $q \equiv -1 \pmod 3$, which requires more work.

So, let q be odd with $3 \mid (q+1)$. According to CHEVIE, we partition $\mathrm{Irr}\,G$ into 31 sets $\mathbf{X}_h$, as listed in Table 11.10.5-A. The Syl_p-decompositions of Table 11.10.5-B allow us to work only with the characters χ_h for $h \in \{1, 2, 3, 5, 6, 7, 9, 11, 15, 22, 27, 30\}$. Note that χ_{27} and χ_{30} are the only regular characters of this set. Let ψ be a reducible Syl_p-vanishing character of G of degree $|G|_p$ whose constituents belong to the above set of h's.

First, assume that χ_{27} is a constituent of ψ. Using the class C_7, we conclude that either χ_{15} or χ_{22} occurs as a constituent of ψ. In the former case, we get $\psi_1 = \chi_5 + \chi_6 + \chi_6 + \chi_7 + \chi_7 + \chi_9 + \chi_{15} + \chi_{27}$. In the latter case, we get $\psi_2 = \chi_1 + \chi_6 + \chi_7 + \chi_9 + \chi_{15} + \chi_{22} + \chi_{27}$.

Next, assume that χ_{30} is a constituent of ψ. Then, with some work we obtain a contradiction.

Finally, applying the Syl_p-decompositions of Table 11.10.5-B to ψ_1, ψ_2, we determine all Syl_p-vanishing characters of G of degree $|G|_p$ (see Table 11.10.5-C).

Lemma 7.11. *The group $G = G_2(q)$ has no reducible p-vanishing character of degree $|G|_p$.*

Proof. Suppose the contrary. As before, we consider here only the case with q odd and $3 \mid (q+1)$. The proof for the other cases is very similar.

(1) Take $\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_{13}$. All these characters do not vanish on the class C_{14}.

(2) Take $\psi_6 = 1_G + \chi_{15} + \chi_{16} + \chi_{22}(k_{22}) + \chi_{27}(k_{27}, l_{27})$. On the classes $C_{23}(i)$ $(i = 1, 2, 3)$, we have

$$1 + Re(\xi_1^{ik_{22}}) + Re(\xi_1^{2ik_{22}}) = Re(\xi_1^{i(k_{27}+l_{27})}) + Re(\xi_1^{i(k_{27}-2l_{27})}) + Re(\xi_1^{i(2k_{27}-l_{27})}).$$

This implies that $q+1$ divides at least one of the integers $k_{27}+l_{27}$, $k_{27}-2l_{27}$, $2k_{27} - l_{27}$, a contradiction.

(3) Take $\psi_7 = 1_G + \chi_{15} + \chi_{16} + \chi_{23}(k_{23})$. On the class $C_{23}(1)$ we have $Re(\xi_1^{k_{23}}) = 1$, a contradiction.

(4) Take $\psi_8 = 1_G + \chi_{22}(k_{22}) + \chi_{24}(k_{24}) + \chi_{27}(k_{27}, l_{27})$. On the classes $C_{23}(i)$ we have

$$Re(\xi_1^{ik_{22}}) + Re(\xi_1^{2ik_{22}}) + Re(\xi_1^{3ik_{24}}) = Re(\xi_1^{i(k_{27}+l_{27})}) + Re(\xi_1^{i(k_{27}-2l_{27})}) + Re(\xi_1^{i(2k_{27}-l_{27})})$$

and on the classes $C_{25}(i)$ we have

$$Re(\xi_1^{ik_{22}}) + Re(\xi_1^{ik_{24}}) + Re(\xi_1^{2ik_{24}}) = Re(\xi_1^{ik_{27}}) + Re(\xi_1^{il_{27}}) + Re(\xi_1^{i(k_{27}-l_{27})}).$$

These conditions lead to a contradiction.

(5) Take $\psi_9 = 1_G + \chi_{22}(k_{22}) + \chi_{25}(k_{25})$. On the classes $C_{23}(i)$ $(i = 1, 2)$ we have $1 + Re(\xi_1^{3ik_{25}}) = Re(\xi_1^{ik_{22}}) + Re(\xi_1^{2ik_{22}})$. This implies that $(q + 1)/2$ divides k_{22}, which is a contradiction.

(6) Take $\psi_{10} = 1_G + \chi_{23}(k_{23}) + \chi_{24}(k_{24})$. On the classes $C_{25}(i)$ $(i = 1, 2)$ we obtain $1 + Re(\xi_1^{ik_{23}}) = Re(\xi_1^{ik_{24}}) + Re(\xi_1^{2ik_{24}})$. This implies that $(q + 1)/2$ divides k_{24}, a contradiction.

(7) Take $\psi_{11} = \chi_4 + \chi_6 + \chi_7 + \chi_{22}(k_{22}) + \chi_{27}(k_{27}, l_{27})$. On the classes $C_{23}(i)$ $(i = 1, 2, 3)$ we obtain

$$1 + Re(\xi_1^{ik_{22}}) + Re(\xi_1^{2ik_{22}}) = Re(\xi_1^{i(k_{27}+l_{27})}) + Re(\xi_1^{i(k_{27}-2l_{27})}) + Re(\xi_1^{i(2k_{27}+l_{27})}).$$

This implies that $q+1$ divides at least one of the numbers $k_{27}+l_{27}$, $k_{27}-2l_{27}$, $2k_{27} - l_{27}$, which contradicts the description of the parameter sets I_h.

(8) Take $\psi_{12} = \chi_4 + \chi_6 + \chi_7 + \chi_{23}(k_{23})$. On the class $C_{23}(1)$ we obtain $Re(\xi_1^{k_{23}}) = 1$, a contradiction.

(9) Take $\psi_{14} = \chi_5 + \chi_6 + \chi_7 + \chi_{15} + \chi_{16} + \chi_{27}(k_{27}, l_{27})$. On the classes $C_{23}(1)$ we obtain $Re(\xi_1^{k_{27}+l_{27}}) + Re(\xi_1^{k_{27}-2l_{27}}) + Re(\xi_1^{2k_{27}+l_{27}}) = 3$, which implies that $(q + 1)$ divides $(k_{27} + l_{27})$, a contradiction.

(10) Take $\psi_{15} = \chi_5 + \chi_6 + \chi_7 + \chi_{24}(k_{24}) + \chi_{27}(k_{27}, l_{27})$. On the classes $C_{23}(i)$ $(i = 1, 2, 3)$ we obtain

$$Re(\xi_1^{i(k_{27}+l_{27})}) + Re(\xi_1^{i(k_{27}-2l_{27})}) + Re(\xi_1^{i(2k_{27}-l_{27})}) = 2 + Re(\xi_1^{3ik_{24}}),$$

which implies that $q + 1$ divides at least one of the numbers $k_{27} + l_{27}$, $k_{27} - 2l_{27}$, $2k_{27} - l_{27}$, which is a contradiction.

(11) Take $\psi_{16} = \chi_5 + \chi_6 + \chi_7 + \chi_{25}(k_{25})$. On the class $C_{23}(1)$ we have $Re(\xi_1^{3k_{25}}) = 1$, a contradiction.

$\square$

Observe that the group $G = G_2(2)'$ has no reducible Syl_2-vanishing characters of degree $|G|_2$. On the other hand, G has six reducible Syl_2-vanishing characters of degree $2|G|_2$. In the notation of [4]:

(1) $\phi_1 = 2\chi_{13}$;
(2) $\phi_2 = 2\chi_{14}$;
(3) $\phi_3 = \chi_{13} + \chi_{14}$;
(4) $\phi_4 = \chi_1 + \chi_3 + \chi_4 + \chi_9 + \chi_{11}$;
(5) $\phi_5 = \chi_1 + \chi_3 + \chi_5 + \chi_8 + \chi_{12}$;
(6) $\phi_6 = \chi_1 + \chi_3 + \chi_6 + \chi_8 + \chi_9$.

The three characters ϕ_1, ϕ_2, ϕ_3 are 2-vanishing, whereas the other three characters are not. This can be easily proved using the character table of G.

7.7. Groups $^2F_4(q^2)$**.** The groups $^2F_4(q^2)$ require a much larger amount of computations. Note that the official distribution of CHEVIE contains only a partial character table of these groups. In this case, our computations are based on the character table kindly provided by F. Himstedt (constructed for [15]). This table is available in CHEVIE format and so we may apply the same argument as in the previous cases.

Let $G = \, ^2F_4(q^2)$, with $q^2 = 2^{2m+1}$ and $m \geq 0$. According to CHEVIE and [15], Irr G is partitioned into 51 sets, as listed in Table 11.11-A. The Syl_p-decompositions of Table 11.11-B allow us to work only with the characters χ_h for $h \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 29, 30, 38, 40, 41, 42, 43, 44, 46, 47, 48, 51\}$. Note that χ_{38}, χ_{40}, χ_{41} and χ_{51} are the only regular characters χ_h with h in this set. Let ψ be a reducible Syl_p-vanishing character of G of degree $|G|_p$ whose constituents χ_h are with h in the above set.

If $(\chi_{51}, \psi) > 0$, we obtain two Syl_p-vanishing characters of G of degree $|G|_p$:

$$\psi_1 \;=\; \chi_2 + \chi_4 + \chi_{10} + \chi_{11} + \chi_{13} + \chi_{17} + \chi_{20} + \chi_{46} + \chi_{47} + \chi_{47} + \chi_{48} + \chi_{51};$$

$$\psi_2 \;=\; \chi_3 + \chi_4 + \chi_{10} + \chi_{12} + \chi_{14} + \chi_{17} + \chi_{19} + \chi_{46} + \chi_{47} + \chi_{48} + \chi_{48} + \chi_{51}.$$

If χ_{38}, χ_{40} or χ_{41} is a constituent of ψ we get a contradiction.

Now, applying the Syl_p-decompositions of Table 11.11-B to the characters ψ_1, ψ_2, we determine all Syl_p-vanishing characters of G of degree $|G|_p$ (see Table 11.11-C).

Lemma 7.12. *The group $G = \, ^2F_4(q^2)$ has no reducible p-vanishing character of degree $|G|_p$.*

Proof. Suppose the contrary. According with Table 11.11-C, there are four cases.

(1) Take $\psi_1 = \chi_2 + \chi_4 + \chi_{10} + \chi_{11} + \chi_{13} + \chi_{17} + \chi_{20} + \chi_{46}(k_{46}) + \chi_{47}(k_{47}) + \chi_{47}(k'_{47}) + \chi_{48}(k_{48}) + \chi_{51}(k_{51}, l_{51})$. Comparing the values at the classes $C_{42}(a)$ and $C_{43}(a)$ $(a = 1, \ldots, 4)$ we obtain

$$Re((\varphi'_8)^{(aqk_{47}\sqrt{2})/2}) + Re((\varphi'_8)^{(ak_{47}(q\sqrt{2}+2)/2)}) + Re((\varphi'_8)^{(aqk'_{47}\sqrt{2})/2}) +$$

$$Re((\varphi'_8)^{(ak'_{47}(q\sqrt{2}+2)/2)}) = 2 + Re((\varphi'_8)^{(aqk_{48}\sqrt{2})/2}) + Re((\varphi'_8)^{(ak_{48}(q\sqrt{2}+2)/2)}).$$

This implies that either $Re((\varphi'_8)^{(qk_{47}\sqrt{2})/2}) = 1$ or $Re((\varphi'_8)^{(k_{47}(q\sqrt{2}+2)/2)}) = 1$. In both the cases, we obtain that $q^2 + q\sqrt{2} + 1$ divides k_{47}, which is a contradiction.

(2) Take ψ_2. Here, we get a contradiction similar to that obtained for ψ_1.

(3) Take $\psi_3 = \chi_2 + \chi_4 + \chi_{10} + \chi_{11} + \chi_{13} + \chi_{17} + \chi_{20} + \chi_{47}(k_{47}) + \chi_{49}(k_{49}, l_{49})$. Comparing the values at the classes $C_{42}(1)$ and $C_{43}(1)$ we obtain

$$Re((\varphi'_8)^{(qk_{47}\sqrt{2})/2}) + Re((\varphi'_8)^{(k_{47}(q\sqrt{2}+2)/2)}) = 2.$$

This implies $Re((\varphi'_8)^{(qk_{47}\sqrt{2})/2}) = 1$, and hence $q^2 + q\sqrt{2} + 1$ divides k_{47}, which is a contradiction.

(4) Take ψ_4. Here, we get a contradiction similar to that obtained for ψ_3.

$\square$

Observe that the group $G = \, ^2F_4(2)'$ has no reducible Syl_2-vanishing characters of degree $|G|_2$. On the other hand, it has four reducible Syl_2-vanishing characters of degree $2|G|_2$. In the notation of [4]:

(1) $\phi_1 = 2\chi_{21}$;
(2) $\phi_2 = 2\chi_{22}$;

(3) $\phi_3 = \chi_{21} + \chi_{22}$;

(4) $\phi_4 = \chi_2 + \chi_3 + \chi_4 + \chi_5 + 2\chi_6 + \chi_{10} + \chi_{11} + \chi_{16} + \chi_{17} + \chi_{20}$.

The three characters ϕ_1, ϕ_2, ϕ_3 are 2-vanishing, whereas the character ϕ_4 is not. This can be easily proved by using the character table of G.

8. THE TRIVIAL CONSTITUENT OF A p-VANISHING CHARACTER

Our next target is to prove that 1_G occurs as a constituent of every reducible p-vanishing character of degree $|G|_p$, if G admits such characters. This turns out to be a rather non-trivial task. We observe that for some groups there is a stronger version of this result, see [28], Proposition 6.2 and Remark following it:

Proposition 8.1. *Let $G = SL(n,q)$, $n > 2$, or $G = E_i(q)$, $i = 6, 7, 8$. Let $\chi \neq St$ be a Syl_p-vanishing character of G of degree $|G|_p$. Then $(\chi, 1_G) = 1$.*

Note that the above statement is not valid for symplectic and unitary groups, as we already seen.

The groups of rank 2 have been analyzed in the previous section. We summarize the results about these groups in the following

Proposition 8.2. *Let G be a Chevalley group of BN-pair rank $\ell = 2$. Let $\chi \neq St$ be a p-vanishing character of G of degree $|G|_p$. Then $(\chi, 1_G) = 1$ and either $G = SL(3,q)$ or $G = Sp(4,q)$ with $7 \mid (q+1)$.*

Proof. See Lemmas 7.5, 7.6, 7.7, 7.8, 7.9, 7.10, 7.11 and 7.12. $\qquad\square$

In order to deal with groups with BN-pair rank $\ell > 2$, we begin with some technical results.

Proposition 8.3. *Let χ be a character of G. Set $\lambda = (\overline{\chi}_{L_J})|_{G_J}$ and $\lambda' = (\overline{\chi}_{L_K})|_{G_K}$ with $J \cup K = I_\ell$.*

(1) *Suppose that $\lambda = St_{G_J}$ and $(\overline{\lambda'}_{T_K}, 1_{T_K})_{T_K} > 0$. Then $\overline{\chi}_{L_J} = St_{L_J}$.*

(2) *Suppose that $\overline{\chi}_T = 1_T$ and that $(\lambda, 1_{G_J})_{G_J} > 0$. Then $(\overline{\chi}_{L_J}, 1_{L_J})_{L_J} = 1$.*

(3) *Suppose that $(\lambda, 1_{G_J})_{G_J} > 0$ and $(\lambda', 1_{G_K})_{G_K} > 0$. In addition, suppose that $\overline{\chi}_T$ is an irreducible character (this holds if χ is a Syl_p-vanishing character of degree $|G|_p$). Then, 1_{L_J} (resp. 1_{L_K}) is a constituent of $\overline{\chi}_{L_J}$ (resp. $\overline{\chi}_{L_K}$) and $\overline{\chi}_T = 1_T$.*

Proof. (1) Note that 1_{T_J} coincides with the truncation of St_{G_J} to T_J. By Lemma 6.7(1), $\overline{\chi}_T$ is a linear character (as so is β_J), and then $\overline{\lambda'}_{T_K}$ is a linear character. So $(\overline{\chi}_T)|_{T_K} = 1_{T_K}$, again by Lemma 6.7. Therefore, $\overline{\chi}_T = 1_T$ by Lemma 6.6.

Since $\lambda = St_{G_J}$, the character $\overline{\chi}_{L_J}$ is irreducible of degree $|L_J|_p = |G_J|_p$, and by Lemma 6.4 this character is St_{L_J}.

(2) By Frobenius reciprocity, $(\lambda, 1_{G_J})_{G_J} = (\overline{\chi}_{L_J}, 1_{G_J}^{L_J})_{L_J} > 0$. Since L_J/G_J is abelian, this implies that $\overline{\chi}_{L_J}$ has an irreducible constituent μ, say, such that $\mu(g) = 1$ for all $g \in G_J$. We show that actually $\mu = 1_{L_J}$. Indeed, the character $\overline{\chi}_T = 1_T$ contains the character $\overline{\mu}_T$, and, obviously, $\overline{\mu}_T = \mu|_T$. Therefore, $\overline{\mu}_T = \mu|_T = 1_T$.

As $L_J = G_J T$, it follows that $\mu = 1_{L_J}$. Furthermore, since $\overline{\chi}_T = 1_T$, we have $(\overline{\chi}_{L_J}, 1_{L_J})_{L_J} = 1$.

(3) First note that if χ is a Syl_p-vanishing character of degree $|G|_p$, then $(\overline{\chi}_T)(1) = (\chi|_U, 1_U) = 1$. Now, since 1_{G_J} is a constituent of λ, it follows that $1_{U \cap L_J}$ is a constituent of $\lambda|_{U \cap L_J}$. This implies that 1_{T_J} is a constituent of $\overline{\lambda}_{T_J}$. By Lemma 6.7, $\overline{\lambda}_{T_J}$ coincides with the linear character $(\overline{\chi}_T)|_{T_J}$. So $\overline{\lambda}_{T_J} = 1_{T_J}$ and, similarly, $\overline{\lambda}_{T_K} = 1_{T_K}$. By Lemma 6.6, $\overline{\chi}_T = 1_T$. Now the result follows from the previous item. $\qquad\square$

Remark 8.4. It follows from (1) that if $\lambda = St_{G_J}$ and $\lambda' = St_{G_K}$ then $\overline{\chi}_{L_J} = St_{L_J}$.

Corollary 8.5. *Let G be a group of BN-pair rank $\ell > 2$, and let J, K be distinct subsets of I_ℓ such that $J \cup K = I_\ell$ and $J \cap K \neq \emptyset$. Let χ be a Syl_p-vanishing character of G and set $\lambda = (\overline{\chi}_{L_J})|_{G_J}$ and $\lambda' = (\overline{\chi}_{L_K})|_{G_K}$. If $\lambda = St_{G_J}$ then $(\lambda', 1_{G_K}) = 0$.*

Proof. Assume $(\lambda', 1_{G_K}) > 0$. By Proposition 8.3(1), $\overline{\chi}_{L_J} = St_{L_J}$. Hence, $\overline{\chi}_{L_{J \cap K}} = St_{L_{J \cap K}}$ and $\overline{\chi}_T = 1_T$. Now, by Proposition 8.3(2) we obtain $(\overline{\chi}_{L_K}, 1_{L_K}) = 1$ and so $\overline{\chi}_{L_{J \cap K}}$ contains a linear character, a contradiction. $\qquad\square$

We may prove now the following

Theorem 8.6. *Let G be a Chevalley group and χ be a p-vanishing character of G of degree $|G|_p$. Then either $\chi = St$ or $(\chi, 1_G) = 1$.*

Proof. We prove this theorem by induction on the BN-pair rank ℓ of G. We already proved that this theorem is true for $\ell = 1$ (Proposition 6.5) and for $\ell = 2$ (Proposition 8.2). Assume $\ell > 2$. Take two subsets J, K of I_ℓ such that $I_\ell = J \cup K$, $J \cap K \neq \emptyset$ and $|K| = 2$.

Set $\lambda = (\overline{\chi}_{L_J})|_{G_J}$ and $\lambda' = (\overline{\chi}_{L_K})|_{G_K}$. Hence, by the induction hypothesis, we have three possibilities:

  (a) $\lambda = St_{G_J}$ and $\lambda' = St_{G_K}$;
  (b) either $\lambda = St_{G_J}$ and $(\lambda', 1_{G_K}) = 1$ or $(\lambda, 1_{G_J}) = 1$ and $\lambda' = St_{G_K}$;
  (c) $(\lambda, 1_{G_J}) = 1$ and $(\lambda', 1_{G_K}) = 1$.

If (a) holds then $\chi = St$ by Propositions 6.15 and 8.3(1). Case (b) is excluded by Corollary 8.5. If (c) holds, then $(\overline{\chi}_{L_J}, 1_{L_J}) = (\overline{\chi}_{L_K}, 1_{L_K}) = 1$ by Proposition 8.3(3). Hence, Proposition 6.15 implies $(\chi, 1_G) = 1$. $\qquad\square$

9. Some Chevalley groups of higher rank

The results of the previous sections are insufficient to start induction. To set up the basis of induction, we additionally need to perform computational work for $H \in \{GL(n, q), n = 4, 5, 6, CSp(6, q)\}$ in spirit of that done for groups of rank 2. Our computations heavily use material available in the CHEVIE package. Observe that CHEVIE does not provide information for groups $H = Spin(n, q)$, $n = 7, 9, 11$ and $H = Spin^+(n, q)$, $n = 8, 10$, so for these groups p-vanishing characters of degree $|H|_p$ remain unknown.

9.1. **Groups $H = GL(4, q)$.** Let ψ be a reducible Syl_p-vanishing character of degree $|H|_p$. According to CHEVIE, we partition $\operatorname{Irr} H$ into 23 sets, see Table 11.12-A. The Syl_p-decompositions of Table 11.12-B allow us to work only with the characters χ_h, where $h \in \{3, 4, 5, 14, 18, 22, 23\}$. Note that χ_{22} and χ_{23} are the only regular characters in this set. By Lemma 3.1, exactly one of these two characters is a constituent of ψ.

(1) Suppose that χ_{22} is a constituent of ψ. Then, by degree reasons, the other constituents of ψ must belong to the set $\{\chi_h \mid h = 4, 5, 14\}$. Looking at the class $C_5(q-1)$ we easily obtain a contradiction.

(2) Suppose that χ_{23} is a constituent of ψ. Then the other constituents of ψ must belong to the set $\{\chi_h \mid h = 3, 4, 5, 14, 18\}$. Looking at the class $C_3(q-1)$, the character χ_{18} must be a constituent of ψ. Now, the character values on the class $C_2(q-1)$ imply that χ_{14} is another constituent of ψ. So, we obtain a unique Syl_p-vanishing character of degree $|H|_p$: $\psi_1 = \chi_5 + \chi_{14} + \chi_{18} + \chi_{23}$.

Now, applying the Syl_p-decompositions of Table 11.12-B to the character ψ_1, we get all Syl_p-vanishing characters of H of degree $|H|_p$ (see Table 11.12-C).

Lemma 9.1. *Let $H = GL(4, q)$. Then H admits reducible p-vanishing characters ψ of degree $|H|_p$ if and only if $3 \mid (q+1)$. In this case H admits exactly $(q-1)(q^2-\delta)/4$ characters ψ with $\delta = \frac{1-(-1)^q}{2}$, where*

$$\psi = \chi_5(k) \cdot (1_H + \chi_{14}(a) + \chi_{18}(0, a) + \chi_{23}((q^2 - 1)b)),$$

$\chi_5(k)$ is a linear character of H, $a = (q+1)/3$ and $b \in \mathbb{Z}_{q^2+1} \setminus \{0\}$.

Proof. Up to linear characters of H we have the following possibilities for ψ:

(1) $\psi_1 = 1_H + \chi_{14}(k_{14,1}) + \chi_{18}(k_{18,1}, k_{18,2}) + \chi_{23}(k_{23,1})$, with $(q-1)$ dividing $2k_{14,1}$, $(2k_{18,1} + k_{18,2})$ and $k_{23,1}$ (so $k_{23,1} = (q-1)k'_{23,1}$). On the classes $C_7(i_1, i_2)$ we obtain $\zeta_1^{(i_2-i_1)k_{18,1}} = 1$, i.e. $k_{18,1} = 0$ and $k_{18,2} = (q-1)k'_{18,2}$. Now, on the classes $C_{12}(i_1, i_2)$ we have $\zeta_1^{(i_1+i_2)k_{14,1}} = 1$, so $k_{14,1} = (q-1)k'_{14,1}$. On the classes $C_{14}(i_1)$ we have

$$1 + Re(\xi_1^{2i_1 k'_{14,1}}) = Re(\xi^{i_1 k'_{18,2}}) + Re(\xi_1^{i_1 k'_{23,1}}),$$

and on the classes $C_{18}(i_1, i_2)$ we have

$$Re(\xi_1^{i_2 k'_{14,1}}) = Re(\xi_1^{i_2 k'_{18,2}}).$$

This holds if and only if 3 divides $q+1$, $\xi_1^{k'_{14,1}}, \xi_1^{k'_{18,2}} \in \{\omega, \omega^2\}$ and $(q+1) \mid k'_{23,1}$.

(2) $\psi_2 = 1_H + \chi_{13}(k_{13,1}) + \chi_{18}(k_{18,1}, k_{18,2})$. On the class $C_{14}(q-1)$ we obtain $Re(\xi_1^{k_{18,2}}) = 1$, i.e. $(q+1) \mid k_{18,2}$, a contradiction. $\qquad \square$

9.2. Groups $H = GL(5, q)$. As in CHEVIE, we partition $\operatorname{Irr} H$ into 43 sets, see Table 11.13-A. The Syl_p-decompositions of Table 11.13-B allow us to work only with the characters χ_h, where $h \in \{5, 6, 7, 23, 24, 30, 36, 41, 42, 43\}$. Note that χ_{41}, χ_{42} and χ_{43} are the only regular characters of this set. Let ψ be a reducible Syl_p-vanishing character of H of degree $|H|_p$.

(1) Let χ_{41} be a constituent of ψ. By degree reasons, the other constituents of ψ must belong to the set $\{\chi_h \mid h = 5, 6, 7, 24, 30, 36\}$. Looking at the class $C_6(q-1)$, we observe that ψ must have at least $q-1$ constituents of type χ_{24}, in contradiction with Lemma 3.10.

(2) Let χ_{42} be a constituent of ψ. Then the other constituents of ψ must belong to the set $\{\chi_h \mid h = 5, 6, 7, 24, 30\}$. Using the class $C_5(q-1)$, we conclude that χ_{24} must be a constituent of ψ. Now, using the class $C_4(q-1)$, with

these constituents we obtain a unique Syl_p-vanishing character of degree $|H|_p$: $\psi_1 = \chi_7 + \chi_{24} + \chi_{30} + \chi_{42}$.

(3) Let χ_{43} be a constituent of ψ. Then the other constituents of ψ must belong to the set $\{\chi_h \mid h = 5, 6, 7, 24, 30, 36\}$. Looking at the classes $C_4(q-1)$ and $C_7(q-1)$, we observe that both χ_{24} and χ_{30} must be constituents of ψ. However, using $C_7(q-1)$, it follows that ψ must admit another constituent of type χ_{24}, which contradicts Lemma 3.10.

Now, applying the Syl_p-decompositions of Table 11.13.B to the character ψ_1, we get all Syl_p-vanishing characters of H of degree $|H|_p$ (see Table 11.13-C).

Lemma 9.2. *Let $H = GL(5,q)$. Then H admits reducible p-vanishing characters ψ of degree $|H|_p$ if and only if $3 \mid (q+1)$. In this case, there are at most $q^2(q-1)/2$ characters*

$$\psi = \chi_7(k) \cdot (1_H + \chi_{24}(0,a) + \chi_{30}(a,0) + \chi_{42}((q^2-1)b,0)),$$

where $\chi_7(k)$ is a linear character of H, $a = (q+1)/3$ and $b \in \mathbb{Z}_{q^2+1} \setminus \{0\}$.

Proof. Modulo linear characters of H we have the following possibilities:

(1) $\psi_1 = 1_H + \chi_{24}(k_{24,1}, k_{24,2}) + \chi_{30}(k_{30,1}, k_{30,2}) + \chi_{42}(k_{42,1}, k_{42,2})$, where $q-1$ must divide $3k_{24,1} + k_{24,2}$, $2k_{30,1} + k_{30,2}$ and $k_{41,1} + k_{42,2}$. On the classes $C_{36}(i_1, i_2)$ we have $\zeta_1^{(i_2 - 3i_1)k_{24,1}} = 1$. So $k_{24,1} = 0$ and $k_{24,2} = (q-1)k'_{24,2}$. Now, on the classes $C_{15}(i_1, i_2)$ we have $\zeta_1^{(i_2 - i_1)k_{30,1}} = 1$, i.e. $k_{30,1} = (q-1)k'_{30,1}$ and $k_{30,2} = 0$. Next, on $C_9(i_1, i_2)$ we have $\zeta_1^{(i_2 - i_1)k_{42,2}} = 1$, i.e. $k_{42,2} = 0$ and $k_{42,1} = (q-1)k'_{42,1}$. On the classes $C_{23}(i_1, i_2)$ we have $Re(\xi_1^{i_2 k'_{24,2}}) = Re(\xi_1^{i_2 k'_{30,1}})$. Finally, consider the classes $C_{30}(i_1, i_2)$ where we have

$$1 + Re(\xi_1^{2i_1 k'_{30,1}}) = Re(\xi_1^{i_1 k'_{30,1}}) + Re(\xi_1^{i_1 k'_{42,1}}).$$

This means that $(q+1) \mid k'_{42,1}$ and $\xi_1^{k'_{30,1}} = \omega^{\pm 1}$. Thus, 3 must divide $q+1$.

(2) $\psi_2 = 1_H + \chi_{24}(k_{24,1}, k_{24,2}) + \chi_{29}(k_{29,1}, k_{29,2})$. On the class $C_{30}(q-1, q-1)$ we obtain $Re(\xi_1^{k_{24,2}}) = 1$, which implies that $q+1$ divides $k_{24,2}$, a contradiction. $\square$

9.3. Groups $H = GL(6,q)$. Let ψ be a reducible Syl_p-vanishing character of H of degree $|H|_p$. We partition $\mathrm{Irr}\, H$ into 112 sets $\mathbf{X}_h$, see Table 11.14-A. The Syl_p-decompositions of Table 11.14-B allow us to work only with the characters χ_h for $h \in \{6, 7, 9, 10, 11, 39, 40, 48, 49, 50, 68, 70, 78, 79, 99, 107, 110, 111, 112\}$. Note that χ_{107}, χ_{110}, χ_{111} and χ_{112} are the only regular characters in this set.

Now, we determine the Syl_p-vanishing characters of H with these constituents:

(1) Suppose that χ_{107} is a constituent of ψ. By degree reasons, the other constituents of ψ must belong to the set $\{\chi_h \mid h = 7, 9, 10, 11, 40, 50\}$. Then, using the class $C_3(q-1)$ we get a contradiction.

(2) Suppose that χ_{110} is a constituent of ψ. Then, by degree reasons, the other constituents of ψ must belong to the set $\{\chi_h \mid h = 6, 7, 9, 10, 11, 39, 40, 48, 49, 50, 68, 70, 79, 99\}$. Then, using the class $C_3(q-1)$, we conclude that some constituent of ψ must be of type χ_{99}. With some computations, we observe that the only possibility remained is the Syl_p-character $\psi_1 = \chi_{11} + \chi_{40} + \chi_{50} + \chi_{68} + \chi_{99} + \chi_{110}$.

(3) Suppose that χ_{111} is a constituent of ψ. By degree reasons, the other constituents of ψ must belong to the set $\{\chi_h \mid h = 6, 7, 9, 10, 11, 39, 40, 48,$ $49, 50, 68, 70, 79\}$. Using the class $C_{11}(q-1)$, we see that either χ_{40} or χ_{50} must be a constituent of ψ. In both the cases, we obtain a contradiction.

(4) Suppose that χ_{112} is a constituent of ψ. Then, by degree reasons, the other constituents of ψ must belong to the set $\{\chi_h \mid h = 6, 7, 9, 10, 11, 39, 40, 48, 49,$ $50, 68, 70, 79, 99\}$. Using the class $C_5(q-1)$, we observe that χ_{39} must be a constituent of ψ. With some computations we again obtain a contradiction.

Now, applying the Syl_p-decompositions of Table 11.14-B to the character ψ_1, we obtain the list of all Syl_p-vanishing characters of H of degree $|H|_p$ (see Table 11.14-C).

Lemma 9.3. *Let $H = GL(6, q)$. Then, both H and $G = H'$ have no reducible p-vanishing character of degree $|H|_p$.*

Proof. Suppose the contrary. Modulo linear characters of H, we have to inspect the following cases.

(1) $\psi_1 = 1_H + \chi_{40}(k_{40,1}) + \chi_{50}(k_{50,1}, k_{50,2}) + \chi_{68}(k_{68,1}, k_{68,2}) + \chi_{99}(k_{99,1}, k_{99,2}) +$ $+ \chi_{110}(k_{110,1}, k_{110,2})$. On the classes $C_{40}((q-1)i'_1)$ $(i'_1 = 1, 2)$ we have $1 + Re(\xi_1^{2i'_1 k_{68,2}}) + Re(\xi_1^{i'_1(k_{110,1}+k_{110,2})}) + Re(\xi_1^{i'_1(k_{110,1}-k_{110,2})}) = Re(\xi_1^{i'_1 k_{40,1}}) +$ $Re(\xi_1^{3i'_1 k_{40,1}}) + Re(\xi_1^{i'_1 k_{50,2}}) +$ $+ Re(\xi_1^{i'_1 k_{99,2}})$. Next, on the classes $C_{39}((q-1)i'_1)$ $(i'_1 = 1, 2)$ we get $1 + Re(\xi_1^{2i'_1 k_{68,2}}) = Re(\xi_1^{i'_1 k_{1,40}}) + Re(\xi_1^{i'_1 k_{50,2}})$. This implies that $q+1$ divides either $k_{40,1}$ or $k_{50,2}$. In both the cases, we have a contradiction.

(2) $\psi_2 = 1_H + \chi_{40}(k_{40,1}) + \chi_{50}(k_{50,1}, k_{50,2}) + \chi_{67}(k_{67,1}, k_{67,2}) + \chi_{110}(k_{110,1}, k_{110,2})$. On the classes $C_{40}((q-1)i'_1)$ $(i'_1 = 1, 2)$ we have

$$Re(\xi_1^{i'_1 k_{40,1}}) + Re(\xi_1^{3i'_1 k_{40,1}}) + Re(\xi_1^{i'_1 k_{50,2}}) = 1 + Re(\xi_1^{i'_1(k_{110,1}+k_{110,2})}) + Re(\xi_1^{i'_1(k_{110,1}-k_{110,2})}).$$

Furthermore, on the classes $C_{39}((q-1)i'_1)$ $(i'_1 = 1, 2)$ we get $1 + Re(\xi_1^{2i'_1 k_{67,1}}) = Re(\xi_1^{i'_1 k_{40,1}}) + Re(\xi_1^{i'_1 k_{50,2}})$. This implies that $q + 1$ divides either $k_{40,1}$ or $k_{50,2}$. In both the cases, we have a contradiction.

(3) $\psi_3 = 1_H + \chi_{38}(k_{38,1}) + \chi_{50}(k_{50,1}, k_{50,2}) + \chi_{68}(k_{68,1}, k_{68,2}) + \chi_{99}(k_{99,1}, k_{99,2})$. On the class $C_{49}(q-1, q-1)$ we have $Re(\xi_1^{k_{50,2}}) = Re(\xi_1^{k_{68,2}})$. On the classes $C_{40}((q-1)i'_2)$ $(i'_2 = 1, 2)$ we have $1 + Re(\xi_1^{i'_2 k_{99,2}}) = Re(\xi_1^{i'_2 k_{50,2}}) + Re(\xi_1^{2i'_2 k_{68,2}})$, which implies $Re(\xi_1^{2k_{68,2}}) = 1$ and $Re(\xi_1^{k_{99,2}}) = Re(\xi_1^{k_{50,2}})$. On the classes $C_{39}((q-1)i'_1)$ $(i'_2 = 1, 2)$ we have $(q^2 + 1)(1 + Re(\xi_1^{2i'_2 k_{68,2}}) = (q^2 + 1)Re(\xi_1^{i'_1 k_{50,2}}) + q^2 Re(\xi_1^{i'_1 k_{38,1}}) + Re(\xi_1^{i'_1 k_{99,1}})$.
So $(q^2 + 2)Re(\xi_1^{i'_1 k_{50,2}}) + q^2 Re(\xi_1^{i'_1 k_{38,1}}) = 2(q^2 + 1)$ which implies $Re(\xi_1^{k_{38,1}}) = 1$, i.e. $(q+1) \mid k_{38,1}$, a contradiction.

(4) $\psi_4 = 1_H + \chi_{40}(k_{40,1}) + \chi_{50}(k_{50,1}, k_{50,2}) + \chi_{68}(k_{68,1}, k_{68,2}) + \chi_{98}(k_{98,1}, k_{98,2})$. On the classes $C_{99}(q-1, q-1)$ we have $Re(\xi_1^{k_{40,1}}) = Re(\xi_1^{k_{68,2}})$, and on the classes $C_{49}(q-1, (q-1)i'_2)$ $(i'_2 = 1, 2)$ we have $q - 1 + Re(\xi_1^{i'_2 k_{50,2}}) = (q-1)Re(\xi_1^{i'_2 k_{40,1}}) + Re(\xi_1^{i'_2 k_{68,2}})$. Hence, $q - 1 + Re(\xi_1^{i'_2 k_{50,2}}) = q Re(\xi_1^{i'_2 k_{40,1}})$, which implies $Re(\xi_1^{k_{40,1}}) = 1$, i.e. $(q+1) \mid k_{40,1}$, a contradiction.

(5) $\psi_5 = 1_H + \chi_{38}(k_{38,1}) + \chi_{50}(k_{50,1}, k_{50,2}) + \chi_{67}(k_{67,1}, k_{67,2})$. On the class $C_{40}(q-1)$ we have $Re(\xi_1^{k_{50,2}}) = 1$, i.e. $(q+1) \mid k_{50,2}$. However, this contradicts the conditions on the parameters.

Since all the conjugacy classes that we have considered belong to $G = H'$, if follows from Proposition 6.2 that G also has no p-vanishing character of degree $|G|_p$. $\hfill\square$

9.4. Groups $H = CSp(6,q)$ with q odd. Let ψ be a reducible Syl_p-vanishing character of degree $|H|_p$. According to CHEVIE, we partition $\mathrm{Irr}\, H$ into 100 sets, see Table 11.15-A. The Syl_p-decompositions of Table 11.15-B allow us to work only with the characters χ_h, where $h \in \{1, 4, 5, 7, 10, 11, 18, 45, 50, 51, 65, 71, 97, 98\}$. Note that the only regular characters belonging to this set are χ_{97} and χ_{98}.

(1) Let χ_{97} be a constituent of ψ. Then, by degree reasons, the other constituents of ψ must belong to the set $\{\chi_h \mid h = 1, 4, 5, 18, 50, 71\}$. Using the class $C_{10}(q-1)$, ψ must have at least one constituent of type χ_{18}. This easily leads to a contradiction.

(2) Let χ_{98} be a constituent of ψ. Then, using the class $C_8(q-1)$, we observe that another constituent of ψ must be either of type χ_{45}, or χ_{50}, or χ_{51}. We get, with some efforts, the following Syl_p-vanishing characters:

$$\psi_1 = \chi_1 + \chi_5 + \chi_{11} + \chi_{18} + \chi_{45} + \chi_{45} + \chi_{45} + \chi_{50} + \chi_{51} + \chi_{65} + \chi_{71} + \chi_{98},$$
$$\psi_2 = \chi_1 + \chi_{11} + \chi_{11} + \chi_{18} + \chi_{45} + \chi_{45} + \chi_{45} + \chi_{45} + \chi_{50} + \chi_{65} + \chi_{71} + \chi_{98},$$
$$\psi_3 = \chi_1 + \chi_5 + \chi_5 + \chi_{18} + \chi_{45} + \chi_{45} + \chi_{50} + \chi_{51} + \chi_{51} + \chi_{65} + \chi_{71} + \chi_{98},$$
$$\psi_4 = \chi_5 + \chi_{10} + \chi_{11} + \chi_{11} + \chi_{18} + \chi_{45} + \chi_{45} + \chi_{45} + \chi_{50} + \chi_{71} + \chi_{98},$$
$$\psi_5 = \chi_5 + \chi_5 + \chi_{10} + \chi_{11} + \chi_{18} + \chi_{45} + \chi_{45} + \chi_{50} + \chi_{51} + \chi_{71} + \chi_{98},$$
$$\psi_6 = \chi_5 + \chi_5 + \chi_5 + \chi_{10} + \chi_{18} + \chi_{45} + \chi_{50} + \chi_{51} + \chi_{51} + \chi_{71} + \chi_{98}.$$

Now, applying the Syl_p-decompositions of Table 11.15-B to the characters ψ_j for $j = 1, \ldots, 6$, we obtain the list of all Syl_p-vanishing characters of H of degree $|H|_p$ (see Table 11.15-C).

Lemma 9.4. *Let $H = CSp(6,q)$, q odd. Then both H and $G = H'$ have no reducible p-vanishing character of degree $|H|_p$.*

Proof. Suppose the contrary. We consider the characters ψ_j described in Table 11.15-C. By Lemma 7.9, a reducible p-vanishing character ψ of G of degree $|G|_p$ must satisfy $(\overline{\psi}_{P_J})|_{G_J} = 1_G + \chi_4(a,b) + \chi_6(c) + \chi_{10}(d)$, where $G_J \cong Sp(4,q)$. So it suffices to consider only the following cases:

(1) Take ψ_j with $j \in \{55, 56\}$. We obtain a contradiction with the value at the class $C_{26}(1,0)$.

(2) Take ψ_j with $j \in \{58, 59, 61, 62, 63, 64, 65, 66, 67, 70, 71, 74, 75\}$. We obtain a contradiction with the values at the class $C_{25}(1,0)$.

(3) Take ψ_{81}. Using the class $C_{70}(1, q-1)$ we have $Re(\xi_1^{k_{85,1}}) = Re(\xi_1^{k_{65,1}})$ and using the class $C_{72}(i_1, 1, q-1)$ we have $Re(\xi_1^{k_{85,2}}) = Re(\xi_1^{k_{41,1}})$. On the classes $C_{74}(i_1, i_2, q-1)$ we have $Re(\xi_1^{(k_{87,1}+k_{87,2})i_1}) + Re(\xi_1^{(k_{87,1}-k_{87,2})i_1}) = Re(\xi_1^{2k_{65,1}i_1}) + Re(\xi_1^{k_{41,1}i_1})$ for all $i_1 = 1, \ldots, \frac{q-1}{2}$. Using the classes $C_{68}(i_1, i_2, q-1)$ we have $Re(\xi_1^{i_1 k_{87,1}}) + Re(\xi_1^{i_1 k_{87,2}}) = Re(\xi_1^{i_1 k_{65,1}}) + Re(\xi_1^{i_1 k_{41,1}})$ for all

$i_1 = 1, \ldots, \frac{q-1}{2}$. Using the classes $C_{38}(i_1, q-1)$ and the previous information, we obtain

$$Re(\xi_1^{(k_{98,1}+k_{98,2}+k_{98,3})i_1}) + Re(\xi_1^{(k_{98,1}-k_{98,2}+k_{98,3})i_1}) +$$

$$+ Re(\xi_1^{(k_{98,1}-k_{98,2}-k_{98,3})i_1}) + Re(\xi_1^{(k_{98,1}+k_{98,2}-k_{98,3})i_1}) =$$

$$= Re(\xi_1^{(2k_{85,1}+k_{85,2})i_1}) + Re(\xi_1^{(2k_{85,1}-k_{85,2})i_1}) + Re(\xi_1^{k_{51,1}i_1}) + Re(\xi^{k_{85,2}i_1})$$

for all $i_1 = 1, \ldots, \frac{q-1}{2}$. Using the classes $C_{37}(i_1, q-1)$ and the previous information, we obtain $1 + Re(\xi_1^{(2k_{85,1}+k_{85,2})i_1}) + Re(\xi_1^{(2k_{85,1}-k_{85,2})i_1}) = Re(\xi_1^{2k_{65,1}i_1}) + Re(\xi_1^{k_{41,1}i_1}) + Re(\xi_1^{3k_{51,1}i_1})$ for all $i_1 = 1, \ldots, \frac{q-1}{2}$. This implies that either $\xi_1^{2k_{65,1}} = 1$, or $\xi_1^{k_{41,1}} = 1$, or $\xi_1^{3k_{51,1}} = 1$. In the first two cases, we get a contradiction with the conditions on the parameters $k_{65,1}$ and $k_{41,1}$. So $\xi_1^{3k_{51,1}} = 1$, i.e. $\xi_1^{k_{51,1}} = \omega^{\pm 1}$. Furthermore, $Re(\xi_1^{(2k_{85,1}+k_{85,2})i_1}) + Re(\xi_1^{(2k_{85,1}-k_{85,2})i_1}) = Re(\xi_1^{2k_{65,1}i_1}) + Re(\xi^{k_{41,1}i_1})$.

First observe that

$$\begin{cases} Re(\xi_1^{k_{85,1}}) = Re(\xi_1^{k_{65,1}}) \\ Re(\xi_1^{k_{85,2}}) = Re(\xi_1^{k_{41,1}}) \\ Re(\xi_1^{2k_{85,1}+k_{85,2}}) + Re(\xi_1^{2k_{85,1}-k_{85,2}}) = Re(\xi_1^{2k_{65,1}}) + Re(\xi_1^{k_{41,1}}) \\ Re(\xi_1^{4k_{85,1}+2k_{85,2}}) + Re(\xi_1^{4k_{85,1}-2k_{85,2}}) = Re(\xi_1^{4k_{65,1}}) + Re(\xi_1^{2k_{41,1}}) \end{cases}.$$

Setting $A = Re(\xi_1^{k_{85,1}})$ and $B = Re(\xi_1^{k_{85,2}})$, we obtain

$$\begin{cases} 2(2A^2 - 1)B = (2A^2 - 1) + B \\ 2(8A^4 - 8A^2 + 1)(2B^2 - 1) = (8A^4 - 8A^2 + 1) + (2B^2 - 1) \end{cases}.$$

Solving this system and remembering that $A \neq \pm 1$, we obtain that either

$$(a) \quad A = \frac{1 \pm \sqrt{5}}{4}, \qquad B = -A,$$

or

$$(b) \quad A = \frac{-1 \pm \sqrt{5}}{4}, \qquad B = A.$$

Now, consider the following equations:

$$\begin{cases} Re(\xi_1^{k_{87,1}}) + Re(\xi_1^{k_{87,2}}) = Re(\xi_1^{k_{65,1}}) + Re(\xi_1^{k_{41,1}}) \\ Re(\xi_1^{k_{87,1}+k_{87,2}}) + Re(\xi_1^{k_{87,1}-k_{87,2}}) = Re(\xi_1^{2k_{65,1}}) + Re(\xi_1^{k_{41,1}}) \\ Re(\xi_1^{2k_{87,1}}) + Re(\xi_1^{2k_{87,2}}) = Re(\xi_1^{2k_{65,1}}) + Re(\xi_1^{2k_{41,1}}) \end{cases}.$$

Setting $X = Re(\xi_1^{k_{87,1}})$ and $Y = Re(\xi_1^{k_{87,2}})$, we obtain in case (a)

$$\begin{cases} 2XY = -1/2 \\ X + Y = 0 \\ X^2 + Y^2 = 2A^2 \end{cases}.$$

So $Y = -X$, $X^2 = \frac{1}{4}$ and $\frac{1}{2} = 2(\frac{1 \pm \sqrt{5}}{4})^2$, an absurd. In case (b) we get

$$\begin{cases} 2XY = -1/2 \\ X + Y = 2A \\ X^2 + Y^2 = 2A^2 \end{cases}.$$

This implies $A^2 = -1/4$, again an absurd.

(4) Take ψ_{82}. Using the classes $C_{70}(1, q-1)$ and $C_{72}(i_1, q+1, q-1)$, we have $Re(\xi_1^{k_{85,1}}) = Re(\xi_1^{k_{65,1}})$ and $Re(\xi_1^{k_{85,2}}) = Re(\xi_1^{k_{41,1}})$. Now, considering also the class $C_{32}(q, q-1)$, we obtain $Re(\xi_1^{k_{85,1}}) + Re(\xi_1^{k_{85,2}}) + Re(\xi_1^{k_{51,1}}) = 1 + Re(\xi_1^{k_{65,1}}) + Re(\xi_1^{k_{41,1}})$, whence $Re(\xi_1^{k_{51,1}}) = 1$, a contradiction.

(5) Take ψ_{83}. Using the class $C_{70}(1, q-1)$ we have $Re(\xi_1^{k_{86,1}}) = Re(\xi_1^{k_{65,1}})$, and looking at the class $C_{72}(i_1, 1, q-1)$ we have $Re(\xi_1^{k_{86,2}}) = Re(\xi_1^{k_{41,1}})$. Using the classes $C_{74}(i_1, i_2, q-1)$ we have $Re(\xi_1^{(k_{87,1}+k_{87,2})i_1}) + Re(\xi_1^{(k_{87,1}-k_{87,2})i_1}) = Re(\xi_1^{2k_{65,1}i_1}) + Re(\xi_1^{k_{41,1}i_1})$ for all $i_1 = 1, 2, 3$. Using the classes $C_{37}(i_1, q-1)$ and the previous information, we obtain $q + qRe(\xi_1^{(2k_{86,1}-k_{86,2})i_1}) + qRe(\xi_1^{(2k_{86,1}+k_{86,2})i_1}) + (q-1)Re(\xi_1^{k_{41,1}i_1}) = qRe(\xi_1^{2k_{65,1}i_1}) + qRe(\xi_1^{3k_{51,1}i_1}) + (2q-1)Re(\xi_1^{k_{51,1}i_1})$ for all $i_1 = 1, \ldots, \frac{q-1}{2}$. This implies that either $\xi_1^{2k_{65,1}} = 1$, or $\xi_1^{k_{51,1}} = 1$, or $\xi_1^{3k_{51,1}} = 1$. In the first two cases we get a contradiction with the conditions on parameters. So $\xi_1^{3k_{51,1}} = 1$, i.e. $\xi_1^{k_{51,1}} = \omega^{\pm 1}$ and $Re(\xi_1^{k_{51,1}}) = -\frac{1}{2}$. Furthermore, $Re(\xi_1^{k_{41,1}}) = Re(\xi_1^{k_{51,1}})$ and

$$(9.1) \qquad Re(\xi_1^{2k_{86,1}+k_{86,2}}) + Re(\xi_1^{2k_{86,1}-k_{86,2}}) = Re(\xi_1^{2k_{65,1}}) + Re(\xi_1^{k_{41,1}}).$$

Using the classes $C_{32}(i_1, q-1)$ we have $Re(\xi_1^{i_1 k_{87,1}}) + Re(\xi_1^{i_1 k_{87,2}}) = Re(\xi_1^{i_1 k_{86,1}}) + Re(\xi_1^{i_1 k_{65,1}}) = 2Re(\xi_1^{i_1 k_{65,1}})$, for all $i_1 = 1, \ldots, \frac{q-1}{2}$. This implies $Re(\xi_1^{k_{87,1}}) = Re(\xi_1^{k_{87,2}}) = Re(\xi_1^{k_{65,1}})$. Using the classes $C_{68}(i_1, j, q-1)$ we have $Re(\xi_1^{k_{87,2}}) = Re(\xi_1^{k_{41,1}})$. Therefore, $Re(\xi_1^{k_{86,1}}) = Re(\xi_1^{k_{86,2}}) = Re(\xi_1^{k_{65,1}}) + Re(\xi_1^{k_{41,1}}) = Re(\xi_1^{k_{51,1}}) = -\frac{1}{2}$. Now, equation (9.1) becomes

$$Re(\xi^{2k_{86,1}+k_{86,2}}) + Re(\xi^{2k_{86,1}-k_{86,2}}) = -1,$$

which has no solution with our values of the parameters.

(6) Take ψ_{84}, ψ_{85}. Using the classes $C_{74}(1, i_2, 0)$, we have $k_{41,1} = q+1$, a contradiction.

(7) Take ψ_{86}, ψ_{87}. Using the classes $C_{72}(i_1, 1, 0)$, we have $k_{41,1} = q+1$, a contradiction.

(8) Take ψ_{92}, ψ_{93}. Using the class $C_{70}(1, 0)$, we have $k_{65,1} = q+1$, a contradiction.

(9) Take ψ_{96}, ψ_{97}. Using the classes $C_{68}(i_1, q, 0)$, we have $k_{65,1} = q+1$, a contradiction.

(10) Take ψ_{78}. Using the classes $C_{70}(q, 0)$ and $C_{72}(i_1, 0, 0)$ we obtain $Re(\xi_1^{k_{41,1}}) = Re(\xi_1^{k_{50,1}}) = Re(\xi_1^{k_{65,1}})$. Using the classes $C_{68}(i_1, q, 0)$ we obtain $Re(\xi_1^{k_{87,1}}) = Re(\xi_1^{k_{87,2}})$, a contradiction.

(11) Take ψ_{79}. Proceed as for ψ_{78}.

(12) Take ψ_{89}. Using the classes $C_{37}(i_1, 0)$ and $C_{74}(i_1, i_2, 0)$ ($i_1 = 0, 1$), we obtain $1 + Re(\xi_1^{k_{52,1}}) = Re(\xi_1^{k_{41,1}}) + Re(\xi_1^{2k_{65,1}})$. So either $q+1$ divides $k_{41,1}$ or $(q+1)/2$ divides $k_{65,1}$. In both the cases, we have a contradiction.

Since all the conjugacy classes that we have considered belong to H', it follows from Proposition 6.2 that also $G = H'$ has no p-vanishing character of degree $|G|_p$. $\square$

10. p-VANISHING CHARACTERS IN HIGH RANK GROUPS

We can now prove that a Chevalley group of rank $\ell \geq 6$ has no p-vanishing character of degree $|G|_p$.

Theorem 10.1. *Let G be a Chevalley group, and χ be a p-vanishing character of G of degree $|G|_p$. Then $\chi = St$ in each of the following cases:*

(1) $SL(n, q)$ *with* $n \geq 6$*;*
(2) $SU(n, q)$ *with* $n \geq 4$*;*
(3) $Sp(2m, q)$ *with either* $m \geq 3$ *if* $7 \mid (q + 1)$ *or* $m \geq 2$ *otherwise;*
(4) $Spin(2m + 1, q)$*, q odd, with* $m \geq 6$*;*
(5) $Spin^+(2m, q)$ *with* $m \geq 6$*;*
(6) $Spin^-(2m, q)$ *with* $m \geq 3$*;*
(7) $E_6(q)$*, $E_7(q)$, $E_8(q)$;*
(8) $^2E_6(q)$*;*
(9) $F_4(q)$*;*
(10) $Spin(2m + 1, q)$*, q odd, with either* $(q + 1, 7) = 1$ *and* $m = 3, 4, 5$ *or* $(q + 1, 3) = 1$ *and* $m = 4, 5$*.*
(11) $Spin^+(2m, q)$ *with* $m = 4, 5$ *and* $(q + 1, 3) = 1$*.*

Proof. We proceed by induction on the BN-pair rank ℓ of the group. Let χ be a p-vanishing character of G of degree $|G|_p$. Take two subsets J, K of I_ℓ with $I_\ell = J \cup K$ and $J \cap K \neq \emptyset$, such that $|K| = 2$. By Theorem 8.6, either $\chi = St$ or $(\chi, 1_G) = 1$. In the latter case, $(\overline{\chi}_L, 1_L) = 1$ for every Levi subgroup L of G. So, we obtain the statement if we are able to find J such that G_J has no reducible p-vanishing characters of degree $|G_J|_p$.

(1) Let $G = SL(n, q)$ with $n \geq 7$. Then we can take J, K as described above, such that $G_J \cong SL(n - 1, q)$ and $G_K \cong SL(3, q)$.

(2) Let $G = SU(n, q)$ with $n \geq 6$. In this case, we can take J, K such that $G_J \cong SU(n - 2, q)$ and $G_K \cong SL(3, q)$

(3) Let $G = Sp(2m, q)$, with either $m \geq 4$ if 7 divides $q + 1$ or $m \geq 3$, otherwise. In this case, we can take J, K such that $G_J = Sp(2m - 2, q)$ and $G_K \cong SL(3, q)$.

(4) Let $G = Spin(2m + 1, q)$ with $m \geq 6$. In this case, we can take J, K such that $G_J = SL(m, q)$ and $G_K \cong Sp(4, q)$. So the result follows from item (1).

(5) Let $G = Spin^+(2m, q)$ with $m \geq 6$. In this case, we can take J, K such that $G_J = SL(m, q)$ and $G_K \cong SL(3, q)$. So the result follows from item (1).

(6) If $G = Spin^-(6, q)$, then $G \cong SU(4, q)$ and so $\chi = St$ by Lemma 7.6. So, assume that $G = Spin^-(2m, q)$ with $m > 3$. Thus, we can take J, K such that $G_J \cong Spin^-(2m - 2, q)$ and $G_K \cong SL(3, q)$

(7) Let $G = E_n(q)$ with $n = 6, 7, 8$. In this case we can take J, K such that $G_J \cong SL(n, q)$ and $G_K \cong SL(3, q)$. So, the result follows from item (1).

(8) Let $G = {}^2E_6(q)$. In this case we can take J, K such that $G_J \cong SU(6, q)$ and $G_K \cong SL(3, q)$.

(9) Let $G = F_4(q)$. In this case we can take J, K such that $G_J \cong Sp(6, q)$ and $G_K \cong \Omega(5, q)$.

(10) Let $G = Spin(2m+1, q)$ and q odd. Assume first that $(q+1, 7) = 1$. In this case, for $m = 3, 4, 5$ we can take $G_J \cong Spin(2m-1, q)$ and $G_K \cong SL(3, q)$. Now, suppose that $(q + 1, 3) = 1$. Hence, for $m = 4, 5$ we can take $G_J \cong SL(m, q)$ and $G_K \cong Sp(4, q)$.

(11) Let $G = Spin^+(2m, q)$ with $m = 4, 5$ and $(q + 1, 3) = 1$. In this case we can take J, K such that $G_J \cong SL(m, q)$ and $G_K \cong SL(3, q)$. So the result follows from Lemmas 9.1 and 9.2

$\square$

11. The tables

The tables below contain information on the Syl_p-vanishing characters and, when they exist, on the reducible p-vanishing characters of degree $St(1)$. (This equals the order of a Sylow p-subgroup U of the group in question.) The tables are organized as follows.

For every quasi-simple Chevalley group G and/or its version H with connected center we assign a subsection in which we provide some properties of the groups G, H and their characters, necessary for understanding our results about these groups tabulated in the same subsection. In some cases we have distinct subsections for the same group H, depending on whether q is odd or even, or congruent to 1 or -1 modulo 3, etc.

First we recall that CHEVIE partitions the conjugacy classes and irreducible characters of H in subsets which we denote here by $\mathbf{C}_h$ and $\mathbf{X}_h$ ($h = 1, 2, \ldots$). Note that the size of every class in $\mathbf{C}_h$ is the same. Similarly, the degrees of the characters in $\mathbf{X}_h$ are the same. (Sometimes the degrees of characters from distinct $\mathbf{X}_h$ may coincide.) We keep the notation of CHEVIE for irreducible characters and conjugacy classes. The symbol $\chi_h(k)$ identifies the irreducible character of the set $\mathbf{X}_h$ associated to the particular element $k \in I_h$ according to CHEVIE. We denote by I_h the set of parameters describing the individual elements of $\mathbf{X}_h$. If it is not necessary to specify the parameter k, we simply write χ_h to indicate any element of the set $\mathbf{X}_h$. Similarly, an individual conjugacy class $C_h(k) \in \mathbf{C}_h$ is the class which corresponds to the parameter $k \in I_h$.

The parameters $k \in I_h$ labeling individual characters $\chi_h(k) \in \mathbf{X}_h$ and classes $C_h(k) \in \mathbf{C}_h$ are represented as elements of a direct product of cyclic groups, with some conditions (note that not necessarily to distinct parameters correspond distinct characters). In the tables we indicate the group I_h belongs to in the columns headed I_h and the exact description of I_h is obtained by the exclusion of the elements written in the columns headed 'exceptions'.

In each subsection we usually give three or four tables headed as Table A, Table B, Table C and Table D (in some cases Table D is omitted).

Tables A provides some information on the irreducible characters of H occurred in Tables B,C,D. Specifically, Tables A list the degrees of the members of the sets $\mathbf{X}_h$, the range of h and we also describe the structure of the character parameter group I_h. In some degenerate case the set $\mathbf{X}_h$ consists of a single element, so no parameter needs to be assigned to the element of the corresponding set I_h. In this case the respective position in the table is marked by $-$, see for instance Table 11.2.2.A. In order to avoid misunderstanding with CHEVIE notation $\chi_h(k)$, in Tables A we use $\deg \chi_h$ to indicate the degree of a character χ_h.

In Tables B, we report the Syl_p-decompositions that we have found and used in our computations, see Sections 5, 7 and 9.

In Tables C we list all the Syl_p-vanishing characters of degree $|H|_p$. The column headed by $\mathbf{v}$ lists, in more friendly notation, the Syl_p-vanishing characters $\psi = \sum \chi_h$, where $\chi_h \in \mathbf{X}_h$ and the sum ranges over the indices from $\mathbf{v}$. Say, if $\mathbf{v} = (1, 4)$ then $\psi = \chi_1 + \chi_4$. The column headed by j is simply the ordering numbers for $\mathbf{v}$; we could write $\mathbf{v}_j$ to be more precise.

Finally, in Tables D, when they exist, we provide the values of the p-vanishing characters ψ of degree $|H|_p$ satisfying the condition $(\psi, 1_H) = 1$. Note that no

information on the character values is lost because all the other p-vanishing characters can be obtained from these by multiplying them with linear characters of H. In addition, we explicitly write down only the values of ψ at those conjugacy classes where ψ differs from the Steinberg character. Unless otherwise stated, the conjugacy classes in Table D are parametrised by the same sets I_h that are used to parametrise the characters in Table A.

To facilitate understanding the tables, as an illustration, in Section 11.1 for groups $GL(2,q)$ and $SL(2,q)$ we write additional comments about the table content.

We use the following notation for some primitive complex roots of the unity: $\omega = \exp(\frac{2\pi i}{3})$, $\xi_1 = \exp(\frac{2\pi i}{q+1})$, $\xi_2 = \exp(\frac{2\pi i}{q^2+1})$, $\varphi'_8 = \exp(\frac{2\pi i}{q^2+\sqrt{2}q+1})$ and $\varphi''_{12} = \exp(\frac{2\pi i}{q^2+\sqrt{3}q+1})$.

11.1. Groups $H = GL(2,q)$.

Table (11.1-A). CHEVIE sets $\mathbf{X}_h$, their degrees and parameters for $GL(2,q)$.

h	$\deg \chi_h$	h	$\deg \chi_h$
1	1	2	q
3	$q+1$	4	$q-1$

h	I_h	exceptions
$1,2$	$k \in \mathbb{Z}_{q-1}$	
3	$(k,l) \in \mathbb{Z}_{q-1}^2$	$(q-1) \mid (k-l)$
4	$k \in \mathbb{Z}_{q^2-1}$	$(q+1) \mid k$

These two tables provide the following information. The irreducible characters of the group $GL(2,q)$ are partitioned into 4 sets $\mathbf{X}_h$. The characters of the set $\mathbf{X}_1$ have degree $\deg \chi_1 = 1$ and are parametrised by the elements of the set $I_1 = \mathbb{Z}_{q-1}$. This means that $\mathbf{X}_1 = \{\chi_1(0), \ldots, \chi_1(q-2)\}$. The characters of the set $\mathbf{X}_2$ have degree q and are parametrised by the elements of $I_2 = \mathbb{Z}_{q-1}$. So, $\mathbf{X}_2 = \{\chi_2(u) \mid u \in \mathbb{Z}_{q-1}\}$. The characters of the set $\mathbf{X}_3$ have degree $q+1$ and are parametrised by the elements of $I_3 = \mathbb{Z}_{q-1} \times \mathbb{Z}_{q-1}$. One more time, we put in evidence that different choices of the parameters $(k,l) \in I_3$ may identify the same irreducible character. Furthermore, the exceptions listed in the third column mean that the character $\chi_3(k,l)$ is not a member of $\mathbf{X}_3$ whenever $(q-1) \mid (k-l)$. It turns out that the set $\mathbf{X}_3$ contains $(q-2)(q-1)/2$ distinct characters. Finally, the characters of the set $\mathbf{X}_4$ have degree $q-1$ and are parametrised by the elements of $I_4 = \mathbb{Z}_{q^2-1}$. In this case, $\chi_4(k)$ does not belong to $\mathbf{X}_4$, whenever $(q+1) \mid k$. (Observe that $(q-1) \mid (k-l)$ is equivalent to $k = l$, but we prefer to keep CHEVIE notation.)

Table (11.1-B). Some Syl_p-decompositions of irreducible characters for $GL(2,q)$.

h	$\mathbf{v}$	h	$\mathbf{v}$
2	$(1,4)$	3	$(1,2)$

In this table we give the following information on the Syl_p-decompositions that we were able to find. The restriction to U of each character χ_2 belonging to the set $\mathbf{X}_2$ can be written as the sum of two summands: the first one is the restriction of any character of the set $\mathbf{X}_1$, the second one is the restriction of any character of the set $\mathbf{X}_4$. Briefly, using the notation of Section 4, $\chi_2 \equiv \chi_1 + \chi_4 \pmod{U}$. Similarly, $\chi_3 \equiv \chi_1 + \chi_2 \pmod{U}$.

Table (11.1-C). Syl_p-vanishing characters of degree $St(1)$ for $GL(2,q)$.

j	$\mathbf{v}$	j	$\mathbf{v}$
1	$(1,4)$	2	(2)

In this table we list all Syl_p-characters ψ_j of H of degree $|H|_p$. For $H = GL(2,q)$ they are of two types: $\psi_1 = \chi_1 + \chi_4$ (i.e. the sum of two irreducible characters: the

first one is any character belonging to the set $\mathbf{X}_1$, the second one is any character belonging to the set $\mathbf{X}_4$); $\psi_2 = \chi_2$ (i.e. any character of the set $\mathbf{X}_2$).

Table (11.1-D). p-vanishing characters of degree $St(1)$ and their values for $GL(2,q)$.

In view of Lemma 7.11 we have the character $\psi = 1_H + \chi_4(k)$, where $k \in I_4$ and $k = (q-1)k'$. This character differs from St only on the following classes:

C_h	St	ψ
$C_4(i)$	-1	$1 - 2Re(\xi_1^{ik'})$

In the above table, we give the values of the reducible p-vanishing character ψ and those of the Steinberg character St of H only at the conjugacy classes where these values differ. The notation for the conjugacy classes, which is similar to the notation for the irreducible characters, is taken according to CHEVIE.

11.2. Groups $U(3,q)$ and $SU(3,q)$.

Table (11.2.1-A). CHEVIE sets $\mathbf{X}_h$, their degrees and parameters for $U(3,q)$.

h	$\deg \chi_h$	h	$\deg \chi_h$	h	$\deg \chi_h$
1	1	2	$q(q-1)$	3	q^3
4	$q^2 - q + 1$	5	$q(q^2 - q + 1)$	6	$(q-1)(q^2 - q + 1)$
7	$(q+1)(q^2 - q + 1)$	8	$(q+1)^2(q-1)$		

h	I_h	exceptions
$1, 2, 3$	$u \in \mathbb{Z}_{q+1}$	
$4, 5$	$(u,v) \in \mathbb{Z}_{q+1} \times \mathbb{Z}_{q+1}$	$(q+1) \mid (u-v)$
6	$(u,v,w) \in \mathbb{Z}_{q+1} \times \mathbb{Z}_{q+1} \times \mathbb{Z}_{q+1}$	$(q+1) \mid (u-v), (u-w), (v-w)$
7	$(v,u) \in \mathbb{Z}_{q+1} \times \mathbb{Z}_{q^2-1}$	$(q-1) \mid u$
8	$u \in \mathbb{Z}_{q^3+1}$	$(q^2 - q + 1) \mid u$

Table (11.2.1-B). Some Syl_p-decompositions of irreducible characters for $U(3,q)$.

h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$
3	$(2,5)$	4	$(1,2)$	5	$(4,6)$	7	$(1,3)$	7	$(4,5)$	8	$(2,2,2,6)$

The above table describes 6 types of Syl_p-decompositions of irreducible characters of $U(3,q)$. For instance, the last column tells us that we have $\chi_8 \equiv \chi_2 + \chi_2' + \chi_2'' + \chi_6 \pmod{U}$. This means that the restriction to U of any character of the set $\mathbf{X}_8$ is the sum of the restrictions of four characters: any 3 characters $\chi_2, \chi_2', \chi_2''$ of the set $\mathbf{X}_2$ (distinct or not) and an arbitrary character χ_6 from $\mathbf{X}_6$.

Table (11.2.1-C). Syl_p-vanishing characters of degree $St(1)$ for $U(3,q)$.

j	$\mathbf{v}$	j	$\mathbf{v}$	j	$\mathbf{v}$	j	$\mathbf{v}$
1	$(1,2,2,6)$	2	$(2,4,6)$	3	$(2,5)$	4	(3)

Note that the above table gives the Syl_p-vanishing character $\psi_1 = \chi_1 + \chi_2 + \chi_2' + \chi_6$. This means that ψ_1 is the sum of 4 characters: χ_1 is any character the set $\mathbf{X}_1$, χ_2, χ_2' are any characters (distinct or not) of $\mathbf{X}_2$, and χ_6 is any character of $\mathbf{X}_6$.

Table (11.2.1-D). p-vanishing characters of degree $St(1)$ and their values for $U(3,q)$ when 3 divides $q+1$.

In view of Lemma 5.5, we have the character $\psi = 1_H + \chi_2(a) + \chi_2(2a) + \chi_6(a,2a,3a)$, where $a = (q+1)/3$. This character differs from St only on the following classes:

C_h	St	ψ
$C_6(k,l,m)$	-1	$1 + 4Re(\omega^{k+l+m}) - 2Re(\omega^{k-l}) - 2Re(\omega^{k-m}) - 2Re(\omega^{l-m})$
$C_8(k)$	-1	$1 - 2Re(\omega^k)$

Table (11.2-2.A). CHEVIE sets $\mathbf{X}_h$, their degrees and parameters for $SU(3,q)$ with the condition that $3 \nmid (q+1)$.

h	$\deg \chi_h$	h	$\deg \chi_h$	h	$\deg \chi_h$	h	$\deg \chi_h$
1	1	2	$q(q-1)$	3	q^3	4	$q^2 - q + 1$
5	$q(q^2 - q + 1)$	6	$(q-1)(q^2 - q + 1)$	7	$(q+1)(q^2 - q + 1)$	8	$(q+1)^2(q-1)$

h	I_h	exceptions
$1,2,3$	$-$	
$4,5$	$n \in \mathbb{Z}_{q+1}$	$(q+1) \mid n$
6	$(n,m) \in \mathbb{Z}_{q+1} \times \mathbb{Z}_{q+1}$	$(q+1) \mid (n-m)$
7	$n \in \mathbb{Z}_{q^2-1}$	$(q-1) \mid n$
8	$n \in \mathbb{Z}_{q^2-q+1}$	$(q^2 - q + 1) \mid n$

Note that each set $\mathbf{X}_1$, $\mathbf{X}_2$ and $\mathbf{X}_3$ consists of a single character.

Table (11.2.2-B). Some Syl_p-decompositions of irreducible characters for $SU(3,q)$ with the condition that $3 \nmid (q+1)$.

h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$
3	$(2,5)$	4	$(1,2)$	5	$(4,6)$	7	$(1,3)$	7	$(4,5)$	8	$(2,2,2,6)$

Table (11.2.2-C). Syl_p-vanishing characters of degree $St(1)$ for $SU(3,q)$ with the condition that $3 \nmid (q+1)$.

j	$\mathbf{v}$	j	$\mathbf{v}$	j	$\mathbf{v}$	j	$\mathbf{v}$
1	$(1,2,2,6)$	2	$(2,4,6)$	3	$(2,5)$	4	(3)

Table (11.2.3-A). CHEVIE sets $\mathbf{X}_h$, their degrees and parameters for $SU(3,q)$ with the condition that $3 \mid (q+1)$.

h	$\deg \chi_h$	h	$\deg \chi_h$	h	$\deg \chi_h$
1	1	2	$q(q-1)$	3	q^3
4	$q^2 - q + 1$	5	$q(q^2 - q + 1)$	6	$(q+1)(q^2 - q + 1)$
$7,8,9$	$\frac{1}{3}(q-1)(q^2 - q + 1)$	10	$(q-1)(q^2 - q + 1)$	$11,\ldots,16$	$\frac{1}{3}(q+1)^2(q-1)$
17	$(q+1)^2(q-1)$				

h	I_h	exceptions
$1,2,3,7,8,9,11,\ldots,16$	$-$	
$4,5$	$n \in \mathbb{Z}_{q+1}$	$(q+1) \mid n$
6	$n \in \mathbb{Z}_{q^2-1}$	$(q-1) \mid n$
10	$(n,m) \in \mathbb{Z}_{q+1} \times \mathbb{Z}_{q+1}$	$(q+1) \mid n, m, (n-m)$
17	$n \in \mathbb{Z}_{q^2-q+1}$	$\frac{q^2 - q + 1}{3} \mid n$

Table (11.2.3-B). Some Syl_p-decompositions of irreducible characters for $SU(3,q)$ with the condition that $3 \mid (q+1)$.

h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$
11	(14)	12	(16)	13	(15)	3	$(2,5)$	4	$(1,2)$
5	$(4,10)$	6	$(1,3)$	6	$(4,5)$	14	$(2,7)$	15	$(2,9)$
16	$(2,8)$	10	$(7,8,9)$	17	$(2,2,2,7,8,9)$				

Note that the characters of the set $\mathbf{X}_{11}$ are Syl_p-equivalent to those of $\mathbf{X}_{14}$. Similarly for the pairs $(\mathbf{X}_{12}, \mathbf{X}_{16})$ and $(\mathbf{X}_{13}, \mathbf{X}_{15})$.

Table (11.2.3-C). Syl_p-vanishing characters of degree $St(1)$ for $SU(3, q)$ with the condition that $3 \mid (q + 1)$.

j	$\mathbf{v}$	j	$\mathbf{v}$	j	$\mathbf{v}$	j	$\mathbf{v}$
1	$(1, 2, 2, 7, 8, 9)$	2	$(1, 2, 2, 10)$	3	$(1, 2, 7, 8, 13)$	4	$(1, 2, 7, 8, 15)$
5	$(1, 2, 7, 9, 12)$	6	$(1, 2, 7, 9, 16)$	7	$(1, 2, 8, 9, 11)$	8	$(1, 2, 8, 9, 14)$
9	$(1, 7, 12, 13)$	10	$(1, 7, 12, 15)$	11	$(1, 7, 13, 16)$	12	$(1, 7, 15, 16)$
13	$(1, 8, 11, 13)$	14	$(1, 8, 11, 15)$	15	$(1, 8, 13, 14)$	16	$(1, 8, 14, 15)$
17	$(1, 9, 11, 12)$	18	$(1, 9, 11, 16)$	19	$(1, 9, 12, 14)$	20	$(1, 9, 14, 16)$
21	$(2, 4, 7, 8, 9)$	22	$(2, 4, 10)$	23	$(2, 5)$	24	(3)
25	$(4, 7, 8, 13)$	26	$(4, 7, 8, 15)$	27	$(4, 7, 9, 12)$	28	$(4, 7, 9, 16)$
29	$(4, 8, 9, 11)$	30	$(4, 8, 9, 14)$				

Table (11.2.3-D). p-vanishing characters of degree $St(1)$ and their values for $SU(3, q)$ with the condition that $3 \mid (q + 1)$.

In view of Lemma 5.7, we have the character $\psi = 1_G + 2\chi_2 + \chi_7 + \chi_8 + \chi_9$. This character differs from St only on the following classes:

C_h	St	ψ
$C_8(a, b)$	-1	$5 - 6Re(\omega^{a-b})$

Here $(a, b) \in I_8 = \mathbb{Z}_{q+1} \times \mathbb{Z}_{q+1}$ with the exceptions: $(q + 1) \mid a, b, (a - b)$.

11.3. **Groups** $^2B_2(q^2)$.

Table (11.3-A). CHEVIE sets $\mathbf{X}_h$, their degrees and parameters for $^2B_2(q^2)$.

h	$\deg \chi_h$	h	$\deg \chi_h$
1	1	2	$\frac{1}{2}\sqrt{2}q(q - 1)(q + 1)$
3	$\frac{1}{2}\sqrt{2}q(q - 1)(q + 1)$	4	q^4
5	$q^4 + 1$	6	$(q - 1)(q^2 - \sqrt{2}q + 1)(q + 1)$
7	$(q - 1)(q^2 + \sqrt{2}q + 1)(q + 1)$		

h	I_h	exceptions
$1, 2, 3, 4$	$-$	
5	$s \in \mathbb{Z}_{q^2-1}$	$(q^2 - 1) \mid s$
6	$k \in \mathbb{Z}_{q^2+\sqrt{2}q+1}$	$(q^2 + \sqrt{2}q + 1) \mid k$
7	$u \in \mathbb{Z}_{q^2-\sqrt{2}q+1}$	$(q^2 - \sqrt{2}q + 1) \mid u$

Table (11.3-B). Some Syl_p-decompositions of irreducible characters for $^2B_2(q^2)$.

h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$
5	$(1, 4)$	4	$(1, 2, 3, 6)$	7	$(2, 2, 3, 3, 6)$

Table (11.3-C). Syl_p-vanishing characters of degree $St(1)$ for $^2B_2(q^2)$.

j	$\mathbf{v}$	j	$\mathbf{v}$
1	$(1, 2, 3, 6)$	2	(4)

Table (11.3-D). p-vanishing characters of degree $St(1)$ and their values for $^2B_2(q^2)$.

In view of Lemma 5.8 we have the character $\psi = 1_G + \chi_2 + \chi_3 + \chi_6(k)$. This character differs from St only on the following classes:

C_h	St	ψ
$C_6(b)$	-1	$3 - 2Re((\varphi_8')^{bk}) - 2Re((\varphi_8')^{q^2 bk})$

11.4. Groups $^2G_2(q^2)$.

Table (11.4-A). CHEVIE sets $\mathbf{X}_h$, their degrees and parameters for $^2G_2(q^2)$.

h	$\deg \chi_h$	h	$\deg \chi_h$
1	1	2	$\frac{1}{6}\sqrt{3}q(q^2-1)(q^2+\sqrt{3}q+1)$
3	$\frac{1}{6}\sqrt{3}q(q^2-1)(q^2-\sqrt{3}q+1)$	4	$\frac{1}{6}\sqrt{3}q(q^2-1)(q^2+\sqrt{3}q+1)$
5	$\frac{1}{6}\sqrt{3}q(q^2-1)(q^2-\sqrt{3}q+1)$	6	$\frac{1}{3}\sqrt{3}q(q-1)(q^2+1)(q+1)$
7	$\frac{1}{3}\sqrt{3}q(q-1)(q^2+1)(q+1)$	8	q^6
9	q^4-q^2+1	10	$q^2(q^4-q^2+1)$
11	$(q^2+1)(q^4-q^2+1)$	12	$(q^2-1)(q^2+\sqrt{3}q+1)(q^2+1)$
13	$(q^2-1)(q^4-q^2+1)$	14	$(q^2-1)(q^2-\sqrt{3}q+1)(q^2+1)$

h	I_h	exceptions
$1,\ldots,10$	$-$	
11	$k \in \mathbb{Z}_{q^2-1}$	$\frac{q^2-1}{2} \mid k$
12	$k \in \mathbb{Z}_{q^2-\sqrt{3}q+1}$	$(q^2-\sqrt{3}q+1) \mid k$
13	$(k,l) \in \mathbb{Z}_{\frac{q^2+1}{2}} \times \mathbb{Z}_2$	$\frac{q^2+1}{4} \mid k$
14	$k \in \mathbb{Z}_{q^2+\sqrt{3}q+1}$	$(q^2-\sqrt{3}q+1) \mid k$

Table (11.4-B). Some decompositions of the restrictions of the irreducible characters on the classes C_1, C_2, C_5 for $^2G_2(q^2)$.

h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$
2	(4)	3	(5)	6	(7)	10	$(9,13)$
11	$(1,8)$	11	$(9,10)$	8	$(1,3,5,6,7,14)$		

Table (11.4-C). Syl_p-vanishing characters of degree $St(1)$ for $^2G_2(q^2)$.

j	$\mathbf{v}$	j	$\mathbf{v}$
1	$(1,3,5,6,7,14)$	2	(8)

Table (11.4-D). p-vanishing characters of degree $St(1)$ and their values for $^2G_2(q^2)$.

In view of Lemma 5.9 we have the character $\psi = 1_G + \chi_3 + \chi_5 + \chi_6 + \chi_7 + \chi_{14}(k)$. This character differs from St only on the following classes:

C_h	St	ψ
$C_{14}(i)$	-1	$5 - 2Re((\varphi_{12}'')^{ik}) - 2Re((\varphi_{12}'')^{(\sqrt{3}q+1)ik}) - 2Re((\varphi_{12}'')^{(\sqrt{3}q+2)ik})$

11.5. Groups $GL(3,q)$ and $SL(3,q)$.

Table (11.5.1-A). CHEVIE sets $\mathbf{X}_h$, their degrees and parameters for $GL(3,q)$.

h	$\deg \chi_h$	h	$\deg \chi_h$
1	1	2	$q(q+1)$
3	q^3	4	q^2+q+1
5	$q(q^2+q+1)$	6	$(q+1)(q^2+q+1)$
7	$(q-1)(q^2+q+1)$	8	$(q-1)^2(q+1)$

h	I_h	exceptions
$1,2,3$	$n \in \mathbb{Z}_{q-1}$	
$4,5$	$(n,m) \in \mathbb{Z}_{q-1} \times \mathbb{Z}_{q-1}$	$(q-1) \mid (n-m)$
6	$(n,m,l) \in \mathbb{Z}_{q-1} \times \mathbb{Z}_{q-1} \times \mathbb{Z}_{q-1}$	$(q-1) \mid (n-m),(n-l),(m-l)$
7	$(m,n) \in \mathbb{Z}_{q-1} \times \mathbb{Z}_{q^2-1}$	$(q+1) \mid n$
8	$n \in \mathbb{Z}_{q^3-1}$	$(q^2+q+1) \mid n$

Table (11.5.1-B). Some Syl_p-decompositions of irreducible characters for $GL(3,q)$.

h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$
3	$(1,7)$	4	$(1,2)$	5	$(2,3)$	5	$(4,7)$	6	$(4,5)$

Table (11.5.1-C). Syl_p-vanishing characters of degree $St(1)$ for $GL(3,q)$.

j	$\mathbf{v}$	j	$\mathbf{v}$
1	$(1,7)$	2	(3)

Table (11.5.1-D). p-vanishing characters of degree $St(1)$ and their values for $GL(3,q)$.

In view of Lemma 7.4 we have the character $\psi = 1_H + \chi_7(q-1,n)$, with $n = (q-1)n'$. This character differs from St only on the following classes:

C_h	St	ψ
$C_7(a,b)$	-1	$1 - 2Re(\xi_1^{bn'})$

Table (11.5.2-A). CHEVIE sets $\mathbf{X}_h$, their degrees and parameters for $SL(3,q)$ with the condition that $3 \mid (q-1)$.

h	$\deg \chi_h$	h	$\deg \chi_h$
1	1	2	$q(q+1)$
3	q^3	4	q^2+q+1
5	$q(q^2+q+1)$	6,7,8	$\frac{1}{3}(q+1)(q^2+q+1)$
9	$(q+1)(q^2+q+1)$	10	$(q-1)(q^2+q+1)$
$11,\ldots,16$	$\frac{1}{3}(q-1)^2(q+1)$	17	$(q-1)^2(q+1)$

h	I_h	exceptions
$1,2,3,6,7,8,11,\ldots,16$	$-$	
$4,5$	$n \in \mathbb{Z}_{q-1}$	$(q-1) \mid n$
9	$(n,m) \in \mathbb{Z}_{q-1}^2$	$(q-1) \mid n, m, (n-m)$
10	$n \in \mathbb{Z}_{q^2-1}$	$(q+1) \mid n$
17	$n \in \mathbb{Z}_{q^2+q+1}$	$\frac{q^2+q+1}{3} \mid n$

Table (11.5.2-B). Some Syl_p-decompositions of irreducible characters for $SL(3,q)$ with the condition that $3 \mid (q-1)$.

h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$
11	(14)	12	(16)	13	(15)	3	$(1,10)$	4	$(1,2)$	5	$(2,3)$
5	$(4,10)$	6	$(2,14)$	7	$(2,16)$	8	$(2,15)$	9	$(4,5)$	17	$(14,15,16)$

Table (11.5.2-C). Syl_p-vanishing characters of degree $St(1)$ for $SL(3,q)$ with the condition that $3 \mid (q-1)$.

j	$\mathbf{v}$	j	$\mathbf{v}$
1	$(1,10)$	2	(3)

Table (11.5.3-A). CHEVIE sets $\mathbf{X}_h$, their degrees and parameters for $SL(3,q)$ with the condition that $3 \nmid (q-1)$.

h	$\deg \chi_h$	h	$\deg \chi_h$
1	1	2	$q(q+1)$
3	q^3	4	q^2+q+1
5	$q(q^2+q+1)$	6	$(q+1)(q^2+q+1)$
7	$(q-1)(q^2+q+1)$	8	$(q-1)^2(q+1)$

h	I_h	exceptions
$1,2,3$	$-$	
$4,5$	$n \in \mathbb{Z}_{q-1}$	$(q-1) \mid n$
6	$(n,m) \in \mathbb{Z}_{q-1}^2$	$(q-1) \mid (n-m)$
7	$n \in \mathbb{Z}_{q^2-1}$	$(q+1) \mid n$
8	$n \in \mathbb{Z}_{q^2+q+1}$	$(q^2+q+1) \mid n$

Table (11.5.3-B). Some Syl_p-decompositions of irreducible characters for $SL(3,q)$ with the condition that $3 \nmid (q-1)$.

h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$
3	$(1,7)$	4	$(1,2)$	5	$(2,3)$	5	$(4,7)$	6	$(4,5)$

Table (11.5.3-C). Syl_p-vanishing characters of degree $St(1)$ for $SL(3,q)$ with the condition that $3 \nmid (q-1)$.

j	$\mathbf{v}$	j	$\mathbf{v}$
1	$(1,7)$	2	(3)

11.6. Groups $U(4,q)$.

Table (11.6-A). CHEVIE sets $\mathbf{X}_h$, their degrees and parameters for $U(4,q)$.

h	$\deg \chi_h$	h	$\deg \chi_h$
1	q^6	2	$q^3(q^2-q+1)$
3	$q^2(q^2+1)$	4	$q(q^2-q+1)$
5	1	6	$(q-1)(q^2+1)q^3$
7	$q(q^2+1)(q-1)^2$	8	$(q-1)(q^2+1)$
9	$q^2(q^2+1)(q^2-q+1)$	10	$q(q^2-q+1)(q^2+1)$
11	$q(q^2-q+1)(q^2+1)$	12	$(q^2+1)(q^2-q+1)$
13	$q^2(q^2-q+1)(q+1)^2$	14	$(q^2-q+1)(q+1)^2$
15	$q(q-1)(q^2+1)(q^2-q+1)$	16	$(q-1)(q^2+1)(q^2-q+1)$
17	$q(q+1)(q^2-q+1)(q^2+1)$	18	$(q+1)(q^2-q+1)(q^2+1)$
19	$(q^2-q+1)(q^2+1)(q-1)^2$	20	$(q-1)(q+1)(q^2+1)(q^2-q+1)$
21	$(q^2+1)(q^2-q+1)(q+1)^2$	22	$(q^2+1)(q-1)^2(q+1)^2$
23	$(q-1)(q^2-q+1)(q+1)^3$		

h	I_h	exceptions
$1 \ldots, 5$	$k_1 \in \mathbb{Z}_{q+1}$	
$6, \ldots, 12$	$(k_1, k_2) \in \mathbb{Z}_{q+1} \times \mathbb{Z}_{q+1}$	$(q+1) \mid (k_1 - k_2)$
$13, 14$	$k_1 \in \mathbb{Z}_{q^2-1}$	$(q-1) \mid k_1$
$15, 16$	$(k_1, k_2, k_3) \in \mathbb{Z}_{q+1} \times \mathbb{Z}_{q+1} \times \mathbb{Z}_{q+1}$	$(q+1) \mid (k_i - k_j), i \neq j$
$17, 18$	$(k_1, k_2) \in \mathbb{Z}_{q+1} \times \mathbb{Z}_{q^2-1}$	$(q^2-1) \mid (q-1)k_1 - k_2$
		$(q-1) \mid k_2$
19	$(k_1, k_2, k_3, k_4) \in \mathbb{Z}_{q+1}^4$	$(q+1) \mid (k_i - k_j), i \neq j$
20	$(k_1, k_2, k_3) \in \mathbb{Z}_{q^2-1} \times \mathbb{Z}_{q+1} \times \mathbb{Z}_{q+1}$	$(q-1) \mid k_1$
		$(q+1) \mid (k_2 - k_3)$
21	$(k_1, k_2) \in \mathbb{Z}_{q^2-1} \times \mathbb{Z}_{q^2-1}$	$(q-1) \mid k_i$
		$(q^2-1) \mid qk_1 + k_2$
		$(q^2-1) \mid (k_1 - k_2)$
22	$(k_1, k_2) \in \mathbb{Z}_{q^3+1} \times \mathbb{Z}_{q+1}$	$(q^2-q+1) \mid k_1$
23	$k_1 \in \mathbb{Z}_{q^4-1}$	$(q^3+q-q^2-1), (q^2+1) \mid k_1$

Table (11.6-B). Some Syl_p-decompositions of irreducible characters for $U(4,q)$.

h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$
10	(11)	1	$(2,6)$	3	$(8,12)$	4	$(5,8)$	6	$(7,15)$	9	$(3,6)$
9	$(11,15)$	9	$(12,20)$	11	$(2,4)$	11	$(3,7)$	11	$(12,16)$	13	$(14,23)$
15	$(16,19)$	16	$(7,8)$	17	$(9,11)$	17	$(18,20)$	18	$(11,12)$	20	$(6,8)$
20	$(15,16)$	21	$(13,14)$	21	$(17,18)$	22	$(7,7,7,19)$	14	$(5,5,8,8,12)$		

Table (11.6-C). Syl_p-vanishing characters of degree $St(1)$ for $U(4,q)$.

j	$\mathbf{v}$	j	$\mathbf{v}$	j	$\mathbf{v}$	j	$\mathbf{v}$	j	$\mathbf{v}$
1	(1)	2	$(2,6)$	3	$(2,7,7,8,19)$	4	$(2,7,15)$	5	$(2,7,16,19)$

11.7. **Groups** $U(5,q)$.

Table (11.7-A). CHEVIE sets $\mathbf{X}_h$, their degrees and parameters for $U(5,q)$.

h	$\deg \chi_h$	h	$\deg \chi_h$
1	q^{10}	2	$q^6(q-1)(q^2+1)$
5	$q^2(q^4-q^3+q^2-q+1)$	8	$q^6(q^4-q^3+q^2-q+1)$
9	$q^3(q^2-q+1)(q^4-q^3+q^2-q+1)$	15	$q^2(q-1)(q^2+1)(q^4-q^3+q^2-q+1)$
16	$q(q-1)(q^2+1)(q^4-q^3+q^2-q+1)$	19	$q^3(q-1)(q^2+1)(q^4-q^3+q^2-q+1)$
20	$q(q^2+1)(q^4-q^3+q^2-q+1)(q-1)^2$	21	$(q-1)(q^2+1)(q^4-q^3+q^2-q+1)$
31	$q(q-1)(q^2-q+1)(q^4-q^3+q^2-q+1)(q^2+1)$	32	$(q-1)(q^2-q+1)(q^4-q^3+q^2-q+1)(q^2+1)$
37	$(q^2-q+1)(q^4-q^3+q^2-q+1)(q^2+1)(q-1)^2$		

h	I_h	exceptions
$1,2,5$	$k_1 \in \mathbb{Z}_{+1}$	
$8,9,15,16$	$(k_1,k_2) \in \mathbb{Z}_{q+1}^2$	$(q+1) \mid k_1-k_2$
$19,20,21$	$(k_1,k_2,k_3) \in \mathbb{Z}_{q+1}^3$	$(q+1) \mid k_i-k_j, i \neq j$
$31,32$	$(k_1,k_2,k_3,k_4) \in \mathbb{Z}_{q+1}^4$	$(q+1) \mid k_i-k_j, i \neq j$
37	$(k_1,k_2,k_3,k_4,k_5) \in \mathbb{Z}_{q+1}^5$	$(q+1) \mid k_i-k_j, i \neq j$

Table (11.7-B). Some Syl_p-decompositions of irreducible characters for $U(5,q)$.

h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$
26	(27)	1	$(2,8)$	3	$(11,16)$	4	$(6,17)$	8	$(9,19)$	9	$(5,15)$
10	$(3,5)$	10	$(16,17)$	10	$(21,28)$	11	$(12,21)$	12	$(6,7)$	13	$(3,8)$
13	$(14,19)$	13	$(15,25)$	14	$(2,4)$	14	$(3,9)$	14	$(10,15)$	14	$(16,27)$
14	$(17,23)$	15	$(16,20)$	17	$(5,11)$	17	$(18,21)$	18	$(5,12)$	19	$(20,31)$
22	$(13,14)$	22	$(23,33)$	23	$(2,6)$	23	$(15,16)$	24	$(17,18)$	25	$(10,19)$
25	$(27,31)$	25	$(28,38)$	27	$(9,11)$	27	$(10,20)$	27	$(15,17)$	27	$(28,32)$
28	$(16,18)$	29	$(30,42)$	31	$(32,37)$	32	$(20,21)$	33	$(13,17)$	33	$(25,27)$
33	$(34,38)$	34	$(14,18)$	34	$(23,24)$	34	$(27,28)$	35	$(36,40)$	38	$(19,21)$
38	$(31,32)$	39	$(22,24)$	39	$(29,30)$	39	$(33,34)$	41	$(35,36)$	36	$(2,6,16,21)$
40	$(20,20,20,37)$										

Table (11.7-C). Syl_p-vanishing characters of degree $St(1)$ for $U(5,q)$.

j	$\mathbf{v}$	j	$\mathbf{v}$	j	$\mathbf{v}$
1	$(2,5,15,19)$	2	$(2,5,15,20,20,21,37)$	3	$(2,5,15,20,31)$
4	$(2,5,15,20,32,37)$	5	$(2,5,16,19,20)$	6	$(2,5,16,20,20,20,21,37)$
7	$(2,5,16,20,20,31)$	8	$(2,5,16,20,20,32,37)$	9	$(2,5,16,21,40)$
10	$(2,9,19)$	11	$(2,9,20,20,21,37)$	12	$(2,9,20,31)$
13	$(2,9,20,32,37)$	14	$(2,8)$	15	(1)

11.8. **Groups** $Sp(4,q)$.

Table (11.8.1-A). CHEVIE sets $\mathbf{X}_h$, their degrees and parameters for $Sp(4,q)$ with q even.

h	$\deg \chi_h$	h	$\deg \chi_h$	h	$\deg \chi_h$
1	1	2	$\frac{1}{2}q(q+1)^2$	$3,4$	$\frac{1}{2}q(q^2+1)$
5	$\frac{1}{2}q(q-1)^2$	6	q^4	7	$(q+1)(q^2+1)$
8	$q(q+1)(q^2+1)$	9	$(q-1)(q^2+1)$	10	$q(q-1)(q^2+1)$
11	$(q+1)(q^2+1)$	12	$q(q+1)(q^2+1)$	13	$(q-1)(q^2+1)$
14	$q(q-1)(q^2+1)$	15	$(q+1)^2(q^2+1)$	$16,17$	$(q-1)(q+1)(q^2+1)$
18	$(q-1)^2(q+1)^2$	19	$(q-1)^2(q^2+1)$		

h	I_h	exceptions
$1, \ldots, 6$	$-$	
$7, 8, 11, 12$	$k \in \mathbb{Z}_{q-1}$	$(q-1) \mid k$
$9, 10, 13, 14$	$k \in \mathbb{Z}_{q+1}$	$(q+1) \mid k$
15	$(k, l) \in \mathbb{Z}_{q-1}^2$	$(q-1) \mid k \pm l$
16	$k \in \mathbb{Z}_{q^2-1}$	$(q \pm 1) \mid k$
17	$(k, l) \in \mathbb{Z}_{q-1} \times \mathbb{Z}_{q+1}$	$(q-1) \mid k, (q+1) \mid l$
18	$k \in \mathbb{Z}_{q^2+1}$	$(q^2+1) \mid k$
19	$(k, l) \in \mathbb{Z}_{q+1}^2$	$(q+1) \mid k \pm l$

Table (11.8.1-B). Some Syl_p-decompositions of irreducible characters for $Sp(4, q)$ with q even.

h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$
8	$(7, 17)$	10	$(9, 19)$	12	$(11, 16)$	14	$(13, 19)$	15	$(7, 8)$
15	$(11, 12)$	16	$(13, 14)$	17	$(9, 10)$	7	$(1, 2, 3)$	11	$(1, 2, 4)$
6	$(1, 9, 13, 19)$	6	$(3, 5, 9, 19)$	6	$(4, 5, 13, 19)$				

Table (11.8.1-C). Syl_p-vanishing characters of degree $St(1)$ for $Sp(4, q)$ with q even.

j	$\mathbf{v}$	j	$\mathbf{v}$	j	$\mathbf{v}$	j	$\mathbf{v}$
1	$(1, 9, 13, 19)$	2	$(1, 9, 14)$	3	$(1, 10, 13)$	4	$(3, 5, 9, 19)$
5	$(3, 5, 10)$	6	$(4, 5, 13, 19)$	7	$(4, 5, 14)$	8	(6)

Table (11.8.2-A). CHEVIE sets $\mathbf{X}_h$ and their degrees for $Sp(4, q)$ with q odd.

In this table, for reader's sake, we additionally provide the notation of [24] for the irreducible characters of G.

h	[24]	$\deg \chi_h$	h	[24]	$\deg \chi_h$
1	χ_1	$q^4 - 2q^2 + 1$	2	$-\chi_2$	$q^4 - 1$
3	χ_3	$q^4 + 2q^3 + 2q^2 + 2q + 1$	4	χ_4	$q^4 - 2q^3 + 2q^2 - 2q + 1$
5	$-\chi_5$	$q^4 - 1$	6	$-\chi_6$	$q^3 - q^2 + q - 1$
7	χ_7	$q^4 - q^3 + q^2 - q$	8	χ_8	$q^3 + q^2 + q + 1$
9	χ_9	$q^4 + q^3 + q^2 + q$	10	$-\xi_1$	$q^3 - q^2 + q - 1$
11	$-\xi_1'$	$q^4 - q^3 + q^2 - q$	12	ξ_3	$q^3 + q^2 + q + 1$
13	ξ_3'	$q^4 + q^3 + q^2 + q$	14	$-\xi_{21}$	$(q^4 - 1)/2$
15	$-\xi_{22}$	$(q^4 - 1)/2$	16	ξ_{21}'	$(q^4 - 2q^3 + 2q^2 - 2q + 1)/2$
17	ξ_{22}'	$(q^4 - 2q^3 + 2q^2 - 2q + 1)/2$	18	ξ_{41}	$(q^4 + 2q^3 + 2q^2 + 2q + 1)/2$
19	ξ_{42}	$(q^4 + 2q^3 + 2q^2 + 2q + 1/2$	20	$-\xi_{41}'$	$(q^4 - 1)/2$
21	$-\xi_{42}'$	$(q^4 - 1)/2$	22	$-\Phi_1$	$(q^3 - q^2 + q - 1)/2$
23	$-\Phi_2$	$(q^3 - q^2 + q - 1)/2$	24	$-\Phi_3$	$(q^4 - q^3 + q^2 - q)/2$
25	$-\Phi_4$	$(q^4 - q^3 + q^2 - q)/2$	26	Φ_5	$(q^3 + q^2 + q + 1)/2$
27	Φ_6	$(q^3 + q^2 + q + 1)/2$	28	Φ_7	$(q^4 + q^3 + q^2 + q)/2$
29	Φ_8	$(q^4 + q^3 + q^2 + q)/2$	30	Φ_9	$q^3 + q$
31	θ_1	$(q^4 + q^2)/2$	32	θ_2	$(q^4 + q^2)/2$
33	θ_3	$(q^2 + 1)/2$	34	θ_4	$(q^2 + 1)/2$
35	$-\theta_5$	$(q^4 - q^2)/2$	36	$-\theta_6$	$(q^4 - q^2)/2$
37	$-\theta_7$	$(q^2 - 1)/2$	38	$-\theta_8$	$(q^2 - 1)/2$
39	θ_9	$(q^3 + 2q^2 + q)/2$	40	θ_{10}	$(q^3 - 2q^2 + q)/2$
41	θ_{11}	$(q^3 + q)/2$	42	θ_{12}	$(q^3 + q)/2$
43	θ_{13}	q^4	44	θ_0	1

Table (11.8.2-B). Some Syl_p-decompositions of irreducible characters for $Sp(4, q)$ with q odd.

h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$
2	$(6, 7)$	3	$(8, 9)$	3	$(12, 13)$	3	$(18, 19)$	4	$(16, 17)$	5	$(10, 11)$
5	$(14, 15)$	5	$(20, 21)$	7	$(4, 6)$	9	$(2, 8)$	10	$(22, 23)$	11	$(4, 10)$
11	$(14, 16)$	11	$(15, 17)$	11	$(24, 25)$	12	$(26, 27)$	13	$(5, 12)$	13	$(18, 20)$
13	$(19, 21)$	13	$(28, 29)$	14	$(10, 17)$	14	$(23, 24)$	14	$(35, 38)$	15	$(10, 16)$
15	$(22, 25)$	15	$(36, 37)$	18	$(12, 21)$	18	$(26, 28)$	19	$(12, 20)$	19	$(27, 29)$
20	$(22, 24)$	20	$(35, 37)$	21	$(23, 25)$	21	$(36, 38)$	22	$(37, 40)$	23	$(38, 40)$
24	$(17, 22)$	25	$(16, 23)$	26	$(34, 42)$	27	$(33, 42)$	28	$(15, 26)$	28	$(21, 27)$
28	$(25, 30)$	28	$(31, 41)$	28	$(36, 39)$	29	$(14, 27)$	29	$(20, 26)$	29	$(24, 30)$
29	$(32, 41)$	29	$(35, 39)$	30	$(22, 26)$	30	$(23, 27)$	30	$(39, 40)$	30	$(41, 42)$
31	$(25, 42)$	32	$(24, 42)$	33	$(37, 44)$	34	$(38, 44)$	35	$(24, 40)$	36	$(25, 40)$
39	$(26, 37)$	39	$(27, 38)$	41	$(22, 34)$	41	$(23, 33)$	43	$(31, 35)$	43	$(32, 36)$

h	$\mathbf{v}$
1	$(16, 17, 40, 40, 40, 40)$

Table (11.8.2-C). Syl_p-vanishing characters of degree $St(1)$ for $Sp(4, q)$ with q odd.

j	$\mathbf{v}$	j	$\mathbf{v}$	j	$\mathbf{v}$
1	$(4, 6, 10, 44)$	2	$(4, 6, 22, 23, 44)$	3	$(4, 6, 22, 34, 40)$
4	$(4, 6, 22, 38, 40, 44)$	5	$(4, 6, 23, 33, 40)$	6	$(4, 6, 23, 37, 40, 44)$
7	$(4, 6, 33, 38, 40, 40)$	8	$(4, 6, 34, 37, 40, 40)$	9	$(4, 6, 37, 38, 40, 40, 44)$
10	$(4, 6, 40, 41)$	11	$(4, 10, 40, 42)$	12	$(4, 22, 23, 40, 42)$
13	$(4, 22, 38, 40, 40, 42)$	14	$(4, 23, 37, 40, 40, 42)$	15	$(4, 37, 38, 40, 40, 40, 42)$
16	$(6, 10, 16, 17, 44)$	17	$(6, 11, 44)$	18	$(6, 14, 16, 44)$
19	$(6, 15, 17, 44)$	20	$(6, 16, 17, 22, 23, 44)$	21	$(6, 16, 17, 22, 34, 40)$
22	$(6, 16, 17, 22, 38, 40, 44)$	23	$(6, 16, 17, 23, 33, 40)$	24	$(6, 16, 17, 23, 37, 40, 44)$
25	$(6, 16, 17, 33, 38, 40, 40)$	26	$(6, 16, 17, 34, 37, 40, 40)$	27	$(6, 16, 17, 37, 38, 40, 40, 44)$
28	$(6, 16, 17, 40, 41)$	29	$(6, 16, 23, 24, 44)$	30	$(6, 16, 24, 34, 40)$
31	$(6, 16, 24, 38, 40, 44)$	32	$(6, 16, 34, 35)$	33	$(6, 16, 35, 38, 44)$
34	$(6, 17, 22, 25, 44)$	35	$(6, 17, 25, 33, 40)$	36	$(6, 17, 25, 37, 40, 44)$
37	$(6, 17, 33, 36)$	38	$(6, 17, 36, 37, 44)$	39	$(6, 24, 25, 44)$
40	$(7, 10, 44)$	41	$(7, 22, 23, 44)$	42	$(7, 22, 34, 40)$
43	$(7, 22, 38, 40, 44)$	44	$(7, 23, 33, 40)$	45	$(7, 23, 37, 40, 44)$
46	$(7, 33, 38, 40, 40)$	47	$(7, 34, 37, 40, 40)$	48	$(7, 37, 38, 40, 40, 44)$
49	$(7, 40, 41)$	50	$(10, 16, 17, 40, 42)$	51	$(11, 40, 42)$
52	$(14, 16, 40, 42)$	53	$(15, 17, 40, 42)$	54	$(16, 17, 22, 23, 40, 42)$
55	$(16, 17, 22, 38, 40, 40, 42)$	56	$(16, 17, 23, 37, 40, 40, 42)$	57	$(16, 17, 37, 38, 40, 40, 40, 42)$
58	$(16, 23, 24, 40, 42)$	59	$(16, 23, 32, 40)$	60	$(16, 23, 35, 42)$
61	$(16, 24, 38, 40, 40, 42)$	62	$(16, 32, 38, 40, 40)$	63	$(16, 35, 38, 40, 42)$
64	$(17, 22, 25, 40, 42)$	65	$(17, 22, 31, 40)$	66	$(17, 22, 36, 42)$
67	$(17, 25, 37, 40, 40, 42)$	68	$(17, 31, 37, 40, 40)$	69	$(17, 36, 37, 40, 42)$
70	$(24, 25, 40, 42)$	71	$(24, 31, 40)$	72	$(24, 36, 42)$
73	$(25, 32, 40)$	74	$(25, 35, 42)$	75	$(31, 35)$
76	$(32, 36)$	77	(43)		

Table (11.8.2-D). p-vanishing characters of degree $St(1)$ and their values for $Sp(4,q)$ with q odd and $7 \mid (q+1)$.

In view of Lemma 7.9 we have the character $\psi = 1_G + \chi_4(a,b) + \chi_6(c) + \chi_{10}(d)$. This character differs from St only on the following classes:

C_i	St	ψ
$B_2(i)$	-1	$1 - \beta_{ic}$
$B_4(i,j)$	1	$1 + \beta_{ia}\beta_{jb} + \beta_{ib}\beta_{ja} - \beta_{ic}\beta_{jc} - \beta_{id} - \beta_{jd}$
$B_5(i)$	-1	$1 - \beta_{id}$
$B_6(i)$	$-q$	$q(1 - \beta_{2ic})$
$C_1(i)$	$-q$	$q(1 - \beta_{id})$

Here, $\beta_t = 2Re(\xi_1^t)$.

Table (11.8.3-A). CHEVIE sets $\mathbf{X}_h$ and their degrees for $CSp(4,q)$ with q odd.

Here, for the reader's sake, we also recall the notation of [23] for irreducible characters of $H = CSp(4,q)$.

h	[23]	$\deg \chi_h$	h	[23]	$\deg \chi_h$
1	$X_1(\lambda,\mu)$	$(q+1)^2(q^2+1)$	2	$X_2(\Lambda)$	$q^4 - 1$
3	$X_3(\Lambda,\nu)$	$q^4 - 1$	4	$X_4(\theta)$	$(q^2-1)^2$
5	$X_5(\Lambda,\omega)$	$(q^2+1)(q-1)^2$	6	$X_1(\lambda)$	$(q+1)(q^2+1)$
7	$X_2(\lambda)$	$q(q+1)(q^2+1)$	8	$X_3(\lambda)$	$(q+1)(q^2+1)$
9	$X_4(\lambda)$	$q(q+1)(q^2+1)$	10	$X_5(\omega)$	$(q-1)(q^2+1)$
11	$X_6(\omega)$	$q(q-1)(q^2+1)$	12	$X_7(\Lambda)$	$(q-1)(q^2+1)$
13	$X_8(\Lambda)$	$q(q-1)(q^2+1)$	14	τ_1	$q^2 + 1$
15	τ_2	$q(q^2+1)$	16	τ_3	$q^2(q^2+1)$
17	$\tau_4(\lambda')$	$q^2 - 1$	18	$\tau_5(\lambda')$	$q^2(q^2-1)$
19	θ_1	$q(q+1)^2/2$	20	θ_2	$q(q-1)^2/2$
21	θ_3	$q(q^2+1)/2$	22	θ_4	$q(q^2+1)/2$
23	θ_5	q^4	24	$\theta_0(\lambda)$	1

Table (11.8.3-B). Some Syl_p-decompositions of irreducible characters for $CSp(4,q)$ with q odd.

h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$
1	$(6,7)$	1	$(8,9)$	2	$(10,11)$	3	$(12,13)$	3	$(17,18)$
7	$(3,6)$	8	$(14,15)$	9	$(2,8)$	9	$(15,16)$	11	$(5,10)$
13	$(5,12)$	15	$(10,14)$	15	$(19,20)$	15	$(21,22)$	16	$(2,14)$
16	$(11,15)$	18	$(4,17)$	6	$(14,22,22)$	6	$(19,22,24)$	12	$(17,20,20)$
14	$(17,24,24)$	19	$(17,22,24)$	21	$(17,20,24)$				

h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$
4	$(5,20,20,20,20)$	23	$(5,10,17,20,20,24)$	23	$(5,17,20,20,20,22)$

Table (11.8.3-C). Syl_p-vanishing characters of degree $St(1)$ for $CSp(4,q)$ with q odd.

j	$\mathbf{v}$	j	$\mathbf{v}$	j	$\mathbf{v}$	j	$\mathbf{v}$
1	$(5,10,12,24)$	2	$(5,10,17,20,20,24)$	3	$(5,10,20,21)$	4	$(5,12,20,22)$
5	$(5,17,20,20,20,22)$	6	$(10,13,24)$	7	$(11,12,24)$	8	$(11,17,20,20,24)$
9	$(11,20,21)$	10	$(13,20,22)$	11	(23)		

11.9. Groups $^3D_4(q)$.

Table (11.9.1-A). CHEVIE sets $\mathbf{X}_h$, their degrees and parameters for $^3D_4(q)$ with q even.

h	$\deg \chi_h$	h	$\deg \chi_h$
1	1	2	$q(q^4 - q^2 + 1)$
3	$q^3(q+1)^2(q^4 - q^2 + 1)/2$	4	$q^3(q+1)^2(q^2 - q + 1)^2/2$
5	$q^3(q-1)^2(q^2 + q + 1)^2/2$	6	$q^3(q-1)^2(q^4 - q^2 + 1)/2$
7	$q^7(q^4 - q^2 + 1)$	8	q^{12}
9	$(q+1)(q^2 + q + 1)(q^2 - q + 1)(q^4 - q^2 + 1)$	10	$q^3(q+1)(q^2 - q + 1)(q^2 + q + 1)(q^4 - q^2 + 1)$
11	$(q+1)(q^4 - q^2 + 1)(q^2 - q + 1)^2$	12	$q(q^4 - q^2 + 1)(q+1)^2(q^2 - q + 1)^2$
13	$q^3(q+1)(q^4 - q^2 + 1)(q^2 - q + 1)^2$	14	$(q+1)(q^2 + q + 1)(q^4 - q^2 + 1)(q^2 - q + 1)^2$
15	$q(q+1)(q^2 + q + 1)(q^4 - q^2 + 1)(q^2 - q + 1)^2$	16	$(q^2 + q + 1)(q^4 - q^2 + 1)(q+1)^2(q^2 - q + 1)^2$
17	$(q-1)(q^2 + q + 1)(q^2 - q + 1)(q^4 - q^2 + 1)$	18	$q^3(q-1)(q^2 - q + 1)(q^2 + q + 1)(q^4 - q^2 + 1)$
19	$(q-1)(q+1)(q^2 + q + 1)(q^2 - q + 1)^2(q^4 - q^2 + 1)$	20	$(q-1)(q^4 - q^2 + 1)(q^2 + q + 1)^2$
21	$q(q^4 - q^2 + 1)(q-1)^2(q^2 + q + 1)^2$	22	$q^3(q-1)(q^4 - q^2 + 1)(q^2 + q + 1)^2$
23	$(q-1)(q^2 - q + 1)(q^4 - q^2 + 1)(q^2 + q + 1)^2$	24	$q(q-1)(q^2 - q + 1)(q^4 - q^2 + 1)(q^2 + q + 1)^2$
25	$(q-1)(q+1)(q^2 + q + 1)^2(q^2 - q + 1)(q^4 - q^2 + 1)$	26	$(q-1)^2(q+1)^2(q^2 - q + 1)^2(q^4 - q^2 + 1)$
27	$(q-1)^2(q+1)^2(q^2 + q + 1)^2(q^4 - q^2 + 1)$	28	$(q-1)^2(q+1)^2(q^2 + q + 1)^2(q^2 - q + 1)^2$
29	$(q^2 - q + 1)(q^4 - q^2 + 1)(q-1)^2(q^2 + q + 1)^2$		

h	I_h	exceptions
$1, \ldots, 8$	$-$	
$9, 10$	$k \in \mathbb{Z}_{q-1}$	$(q-1) \mid k$
$11, 12, 13$	$k \in (q^2 + q + 1)$	$(q^2 + q + 1) \mid k$
$14, 15$	$k \in \mathbb{Z}_{q^3 - 1}$	$(q-1) \mid k$
16	$(k,l) \in \mathbb{Z}_{q^3 - 1} \times \mathbb{Z}_{q-1}$	many
$17, 18$	$k \in \mathbb{Z}_{q+1}$	$(q+1) \mid k$
19	$k \in \mathbb{Z}_{(q+1)(q^3 - 1)}$	$(q+1), (q^3 - 1) \mid k$
$20, 21, 22$	$k \in \mathbb{Z}_{q^2 - q + 1}$	$(q^2 - q + 1) \mid k$
$23, 24$	$k \in \mathbb{Z}_{q^3 + 1}$	$(q+1) \mid k$
25	$k \in \mathbb{Z}_{(q-1)(q^3 + 1)}$	$(q-1)(q^3 + 1) \mid k$
26	$(k,l) \in \mathbb{Z}_{q^2 + q + 1}^2$	many
27	$(k,l) \in \mathbb{Z}_{q^2 - q + 1}^2$	many
28	$k \in \mathbb{Z}_{q^4 - q^2 + 1}$	$(q^4 - q^2 + 1) \mid k$
29	$(k,l) \in \mathbb{Z}_{q^3 + 1} \times \mathbb{Z}_{q+1}$	many

Table (11.9.1-B). Some Syl_p-decompositions of irreducible characters for $^3D_4(q)$ with q even.

h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$
10	$(9, 25)$	13	$(11, 19)$	14	$(11, 12)$	15	$(12, 13)$	15	$(14, 19)$
16	$(9, 10)$	16	$(14, 15)$	18	$(17, 29)$	19	$(17, 18)$	22	$(21, 24)$
23	$(20, 21)$	24	$(23, 29)$	25	$(20, 22)$	25	$(23, 24)$	9	$(1, 2, 3, 4)$
27	$(21, 21, 21, 29)$	8	$(1, 17, 20, 21, 29)$	8	$(5, 6, 7, 17, 29)$				

Table (11.9.1-C). Syl_p-vanishing characters of degree $St(1)$ for $^3D_4(q)$ with q even.

j	$\mathbf{v}$	j	$\mathbf{v}$	j	$\mathbf{v}$	j	$\mathbf{v}$
1	$(1, 17, 20, 21, 29)$	2	$(1, 17, 23, 29)$	3	$(1, 17, 24)$	4	$(1, 18, 20, 21)$
5	$(1, 18, 23)$	6	$(2, 5, 6, 20, 21, 29)$	7	$(2, 5, 6, 23, 29)$	8	$(2, 5, 6, 24)$
9	$(5, 6, 7, 17, 29)$	10	$(5, 6, 7, 18)$	11	(8)		

Table (11.9.2-A). CHEVIE sets $\mathbf{X}_h$, their degrees and parameters for $^3D_4(q)$ with q odd.

h	$\deg \chi_h$	h	$\deg \chi_h$
1	1	2	$q(q^4 - q^2 + 1)$
3	$q^3(q+1)^2(q^4 - q^2 + 1)/2$	4	$q^3(q+1)^2(q^2 - q + 1)^2/2$
5	$q^3(q-1)^2(q^2 + q + 1)^2/2$	6	$q^3(q-1)^2(q^4 - q^2 + 1)/2$
7	$q^7(q^4 - q^2 + 1)$	8	q^{12}
9	$(q^2 + q + 1)(q^2 - q + 1)(q^4 - q^2 + 1)$	10	$q(q^2 + q + 1)(q^2 - q + 1)(q^4 - q^2 + 1)$
11	$q^3(q^2 + q + 1)(q^2 - q + 1)(q^4 - q^2 + 1)$	12	$q^4(q^2 + q + 1)(q^2 - q + 1)(q^4 - q^2 + 1)$
13	$(q+1)(q^2 + q + 1)(q^2 - q + 1)(q^4 - q^2 + 1)$	14	$q^3(q+1)(q^2 - q + 1)(q^2 + q + 1)(q^4 - q^2 + 1)$
15	$(q+1)(q^4 - q^2 + 1)(q^2 - q + 1)^2$	16	$q(q^4 - q^2 + 1)(q+1)^2(q^2 - q + 1)^2$
17	$q^3(q+1)(q^4 - q^2 + 1)(q^2 - q + 1)^2$	18	$(q+1)(q^2 + q + 1)(q^4 - q^2 + 1)(q^2 - q + 1)^2$
19	$q(q+1)(q^2 + q + 1)(q^4 - q^2 + 1)(q^2 - q + 1)^2$	20	$(q^2 + q + 1)(q^4 - q^2 + 1)(q+1)^2(q^2 - q + 1)^2$
21	$(q-1)(q^2 + q + 1)(q^2 - q + 1)(q^4 - q^2 + 1)$	22	$q^3(q-1)(q^2 - q + 1)(q^2 + q + 1)(q^4 - q^2 + 1)$
23	$(q-1)(q+1)(q^2 + q + 1)(q^2 - q + 1)^2(q^4 - q^2 + 1)$	24	$(q-1)(q^4 - q^2 + 1)(q+1)^2(q^2 + q + 1)^2$
25	$q(q^4 - q^2 + 1)(q-1)^2(q^2 + q + 1)^2$	26	$q^3(q-1)(q^4 - q^2 + 1)(q^2 + q + 1)^2$
27	$(q-1)(q^2 - q + 1)(q^4 - q^2 + 1)(q^2 + q + 1)^2$	28	$q(q-1)(q^2 - q + 1)(q^4 - q^2 + 1)(q^2 + q + 1)^2$
29	$(q-1)(q+1)(q^2 + q + 1)^2(q^2 - q + 1)(q^4 - q^2 + 1)$	30	$(q-1)^2(q+1)^2(q^2 - q + 1)^2(q^4 - q^2 + 1)$
31	$(q-1)^2(q+1)^2(q^2 + q + 1)^2(q^4 - q^2 + 1)$	32	$(q-1)^2(q+1)^2(q^2 + q + 1)^2(q^2 - q + 1)^2$
33	$(q^2 - q + 1)(q^4 - q^2 + 1)(q-1)^2(q^2 + q + 1)^2$		

h	I_h	exceptions
$1, \ldots, 12$	$-$	
$13, 14$	$k \in \mathbb{Z}_{q-1}$	$(q-1)/2 \mid k$
$15, 16, 17$	$k \in \mathbb{Z}_{q^2+q+1}$	$(q^2 + q + 1) \mid k$
$18, 19$	$k \in \mathbb{Z}_{q^3-1}$	$(q-1), (q^3 - 1)/2 \mid k$
20	$(k,l) \in \mathbb{Z}_{q^3-1} \times \mathbb{Z}_{q-1}$	many
$21, 22$	$k \in \mathbb{Z}_{q+1}$	$(q+1)/2 \mid k$
23	$k \in \mathbb{Z}_{(q+1)(q^3-1)}$	$(q+1), (q^3 - 1), (q+1)(q^3 - 1)/2 \mid k$
$24, 25, 26$	$k \in \mathbb{Z}_{q^2-q+1}$	$(q^2 - q + 1) \mid k$
$27, 28$	$k \in \mathbb{Z}_{q^3+1}$	$(q+1), (q^3 + 1)/2 \mid k$
29	$k \in \mathbb{Z}_{(q-1)(q^3+1)}$	$(q-1), (q^3 + 1), (q-1)(q^3 + 1)/2 \mid k$
30	$(k,l) \in \mathbb{Z}_{q^2+q+1}^2$	many
31	$(k,l) \in \mathbb{Z}_{q^2-q+1}^2$	many
32	$k \in \mathbb{Z}_{q^4-q^2+1}$	$(q^4 - q^2 + 1) \mid k$
33	$(k,l) \in \mathbb{Z}_{q^3+1} \times \mathbb{Z}_{q+1}$	many

Table (11.9.2-B). Some Syl_p-decompositions of irreducible characters for $^3D_4(q)$ with q odd.

h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$
10	$(9, 21)$	11	$(9, 27)$	12	$(10, 28)$	12	$(11, 22)$
13	$(9, 10)$	14	$(11, 12)$	14	$(13, 29)$	17	$(15, 23)$
18	$(9, 11)$	18	$(15, 16)$	19	$(10, 12)$	19	$(16, 17)$
19	$(18, 23)$	20	$(13, 14)$	20	$(18, 19)$	22	$(21, 33)$
23	$(21, 22)$	26	$(25, 28)$	27	$(24, 25)$	28	$(27, 33)$
29	$(24, 26)$	29	$(27, 28)$	31	$(25, 25, 25, 33)$	8	$(1, 21, 24, 25, 33)$
8	$(5, 6, 7, 21, 33)$	32	$(5, 5, 5, 5, 25, 33)$				

Table (11.9.2-C). Syl_p-vanishing characters of degree $St(1)$ for $^3D_4(q)$ with q odd.

j	$\mathbf{v}$	j	$\mathbf{v}$	j	$\mathbf{v}$	j	$\mathbf{v}$
1	$(1, 21, 24, 25, 33)$	2	$(5, 6, 7, 21, 33)$	3	$(2, 5, 6, 24, 25, 33)$	4	$(1, 21, 27, 33)$
5	$(1, 21, 28)$	6	$(1, 22, 24, 25)$	7	$(1, 22, 27)$	8	$(2, 5, 6, 27, 33)$
9	$(2, 5, 6, 28)$	10	$(5, 6, 7, 22)$	11	(8)		

11.10. Groups $G_2(q)$.

Table (11.10.1-A). CHEVIE sets $\mathbf{X}_h$, their degrees and parameters for $G_2(q)$ with q even and $q \equiv 1 \pmod 3$.

h	$\deg \chi_h$	h	$\deg \chi_h$
1	1	2	$q(q+1)^2(q^2+q+1)/6$
3	$q(q+1)^2(q^2-q+1)/2$	4, 5	$q(q^2+q+1)(q^2-q+1)/3$
6	$q(q-1)^2(q^2-q+1)/6$	7	$q(q-1)^2(q^2+q+1)/2$
8, 9	$q(q-1)^2(q+1)^2/3$	10	q^6
11	$(q+1)(q^2-q+1)$	12	$q(q+1)^2(q^2-q+1)$
13	$q^3(q+1)(q^2-q+1)$	14, 16	$(q^2+q+1)(q+1)(q^2-q+1)$
15, 17	$q(q^2+q+1)(q+1)(q^2-q+1)$	18, 20	$(q^2-q+1)(q-1)(q^2+q+1)$
19, 21	$q(q^2-q+1)(q-1)(q^2+q+1)$	22	$(q+1)^2(q^2+q+1)(q^2-q+1)$
23	$(q-1)^2(q^2-q+1)(q^2+q+1)$	24, 25	$(q-1)(q+1)(q^2+q+1)(q^2-q+1)$
26	$(q-1)^2(q+1)^2(q^2-q+1)$	27	$(q+1)^2(q-1)^2(q^2+q+1)$

h	I_h	exceptions
$1, \ldots, 13$	$-$	
14, 15	$k \in \mathbb{Z}_{q-1}$	$(q-1)/3 \mid k$
16, 17	$k \in \mathbb{Z}_{q-1}$	$(q-1) \mid k$
$18, \ldots, 21$	$k \in \mathbb{Z}_{q+1}$	$(q+1) \mid k$
22	$(k,l) \in \mathbb{Z}_{q-1}^2$	$(q-1) \mid k, l, (k \pm l), (2k-l), (k-2l)$
23	$(k,l) \in \mathbb{Z}_{q+1}^2$	$(q+1) \mid k, l, (k \pm l), (2k-l), (k-2l)$
24, 25	$k \in \mathbb{Z}_{q^2-1}$	$(q \pm 1) \mid k$
26	$k \in \mathbb{Z}_{q^2+q+1}$	$(q^2+q+1)/3 \mid k$
27	$k \in \mathbb{Z}_{q^2-q+1}$	$(q^2-q+1) \mid k$

Table (11.10.1-B). Some Syl_p-decompositions of irreducible characters for $G_2(q)$ with q even and $q \equiv 1 \pmod 3$.

h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$
8	(9)	13	$(11, 24)$	14	$(11, 12)$	15	$(12, 13)$	15	$(14, 24)$
17	$(16, 25)$	19	$(18, 23)$	21	$(20, 23)$	22	$(14, 15)$	22	$(16, 17)$
24	$(18, 19)$	25	$(20, 21)$	12	$(2, 3, 9)$	10	$(1, 18, 20, 23)$	16	$(1, 2, 3, 5)$

Table (11.10.1-C). Syl_p-vanishing characters of degree $St(1)$ for $G_2(q)$ with q even and $q \equiv 1 \pmod 3$.

j	$\mathbf{v}$	j	$\mathbf{v}$	j	$\mathbf{v}$	j	$\mathbf{v}$
1	$(1, 18, 20, 23)$	2	$(1, 18, 21)$	3	$(1, 19, 20)$	4	$(4, 6, 7, 18, 23)$
5	$(4, 6, 7, 19)$	6	$(5, 6, 7, 20, 23)$	7	$(5, 6, 7, 21)$	8	(10)

Table (11.10.2-A). CHEVIE sets $\mathbf{X}_h$, their degrees and parameters for $G_2(q)$ with q even and $q \equiv -1 \pmod 3$.

h	$\deg \chi_h$	h	$\deg \chi_h$
1	1	2	$q(q+1)^2(q^2+q+1)/6$
3	$q(q+1)^2(q^2-q+1)/2$	4, 5	$q(q^2+q+1)(q^2-q+1)/3$
6	$q(q-1)^2(q^2-q+1)/6$	7	$q(q-1)^2(q^2+q+1)/2$
8, 9	$q(q-1)^2(q+1)^2/3$	10	q^6
11	$(q-1)(q^2+q+1)$	12	$q(q-1)^2(q^2+q+1)$
13	$q^3(q-1)(q^2+q+1)$	14, 16	$(q^2+q+1)(q+1)(q^2-q+1)$
15, 17	$q(q^2+q+1)(q+1)(q^2-q+1)$	18, 20	$(q^2-q+1)(q-1)(q^2+q+1)$
19, 21	$q(q^2-q+1)(q-1)(q^2+q+1)$	22	$(q+1)^2(q^2+q+1)(q^2-q+1)$
23	$(q-1)^2(q^2-q+1)(q^2+q+1)$	24, 25	$(q-1)(q+1)(q^2+q+1)(q^2-q+1)$
26	$(q-1)^2(q+1)^2(q^2-q+1)$	27	$(q+1)^2(q-1)^2(q^2+q+1)$

h	I_h	exceptions
$1, \ldots, 13$	$-$	
$14, \ldots, 17$	$k \in \mathbb{Z}_{q-1}$	$(q-1) \mid k$
$18, 19$	$k \in \mathbb{Z}_{q+1}$	$(q+1) \mid k$
$20, 21$	$k \in \mathbb{Z}_{q+1}$	$(q+1)/3 \mid k$
22	$(k,l) \in \mathbb{Z}_{q-1}^2$	$(q-1) \mid k, l, (k \pm l), (2k-l), (k-2l)$
23	$(k,l) \in \mathbb{Z}_{q+1}^2$	$(q+1) \mid k, l, (k \pm l), (2k-l), (k-2l)$
$24, 25$	$k \in \mathbb{Z}_{q^2-1}$	$(q \pm 1) \mid k$
26	$k \in \mathbb{Z}_{q^2+q+1}$	$(q^2+q+1) \mid k$
27	$\mathbb{Z}_{q^2-q+1}$	$(q^2-q+1)/3 \mid k$

Table (11.10.2-B). Some Syl_p-decompositions of irreducible characters for $G_2(q)$ with q even and $q \equiv -1 \pmod 3$.

h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$
8	(9)	13	$(12, 21)$	15	$(14, 24)$	17	$(16, 25)$	19	$(18, 23)$
20	$(11, 12)$	21	$(20, 23)$	22	$(14, 15)$	22	$(16, 17)$	24	$(18, 19)$
25	$(11, 13)$	25	$(20, 21)$	4	$(1, 9, 11)$	12	$(6, 7, 9)$	16	$(1, 2, 3, 5)$

h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$
14	$(1, 1, 2, 3, 9, 11)$	10	$(1, 6, 7, 9, 11, 18, 23)$	27	$(6, 6, 6, 7, 7, 7, 9, 9, 9, 23)$

Table (11.10.2-C). Syl_p-vanishing characters of degree $St(1)$ for $G_2(q)$ with q even and $q \equiv -1 \pmod 3$.

j	$\mathbf{v}$	j	$\mathbf{v}$	j	$\mathbf{v}$
1	$(1, 6, 7, 8, 11, 18, 23)$	2	$(1, 6, 7, 8, 11, 19)$	3	$(1, 6, 7, 9, 11, 18, 23)$
4	$(1, 6, 7, 9, 11, 19)$	5	$(1, 11, 12, 18, 23)$	6	$(1, 11, 12, 19)$
7	$(1, 18, 20, 23)$	8	$(1, 18, 21)$	9	$(1, 19, 20)$
10	$(4, 6, 7, 18, 23)$	11	$(4, 6, 7, 19)$	12	$(5, 6, 6, 7, 7, 8, 11, 23)$
13	$(5, 6, 6, 7, 7, 9, 11, 23)$	14	$(5, 6, 7, 11, 12, 23)$	15	$(5, 6, 7, 20, 23)$
16	$(5, 6, 7, 21)$	17	(10)		

Table (11.10.3-A). CHEVIE sets $\mathbf{X}_h$, their degrees and parameters for $G_2(q)$ with q odd and $q \equiv 0 \pmod 3$.

h	$\deg \chi_h$	h	$\deg \chi_h$
1	1	2	$q(q+1)^2(q^2+q+1)/6$
3	$q(q+1)^2(q^2-q+1)/2$	4, 5	$q(q^2+q+1)(q^2-q+1)/3$
6	$q(q-1)^2(q^2-q+1)/6$	7	$q(q-1)^2(q^2+q+1)/2$
8, 9	$q(q-1)^2(q+1)^2/3$	10	q^6
11	$(q^2+q+1)(q^2-q+1)$	12, 13	$q(q^2+q+1)(q^2-q+1)$
14	$q^2(q^2+q+1)(q^2-q+1)$	15, 17	$(q^2+q+1)(q+1)(q^2-q+1)$
16, 18	$q(q^2+q+1)(q+1)(q^2-q+1)$	19, 21	$(q^2-q+1)(q-1)(q^2+q+1)$
20, 22	$q(q^2-q+1)(q-1)(q^2+q+1)$	23	$(q+1)^2(q^2+q+1)(q^2-q+1)$
24	$(q-1)^2(q^2-q+1)(q^2+q+1)$	25, 26	$(q-1)(q+1)(q^2+q+1)(q^2-q+1)$
27	$(q-1)^2(q+1)^2(q^2-q+1)$	28	$(q+1)^2(q-1)^2(q^2+q+1)$

h	I_h	exceptions
$1, \ldots 14$	$-$	
$15, \ldots, 18$	$k \in \mathbb{Z}_{q-1}$	$(q-1)/2 \mid k$
$19, \ldots, 22$	$k \in \mathbb{Z}_{q+1}$	$(q+1)/2 \mid k$
23	$(k,l) \in \mathbb{Z}_{q-1}^2$	$(q-1) \mid k, l, (k \pm l), (2k-l), (k-2l)$
24	$(k,l) \in \mathbb{Z}_{q+1}^2$	$(q+1) \mid k, l, (k \pm l), (2k-l), (k-2l)$
$25, 26$	$k \in \mathbb{Z}_{q^2-1}$	$(q \pm 1) \mid k$
27	$k \in \mathbb{Z}_{q^2+q+1}$	$(q^2+q+1) \mid k$
28	$k \in \mathbb{Z}_{q^2-q+1}$	$(q^2-q+1) \mid k$

Table (11.10.3-B). Some Syl_p-decompositions of irreducible characters for $G_2(q)$ with q odd and $q \equiv 0 \pmod 3$.

h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$
12	$(11,19)$	13	$(11,21)$	14	$(12,22)$	14	$(13,20)$	15	$(11,13)$
16	$(12,14)$	16	$(15,25)$	17	$(11,12)$	18	$(13,14)$	18	$(17,26)$
20	$(19,24)$	22	$(21,24)$	23	$(15,16)$	23	$(17,18)$	25	$(19,20)$
26	$(21,22)$	10	$(1,19,21,24)$						

Table (11.10.3-C). Syl_p-vanishing characters of degree $St(1)$ for $G_2(q)$ with q odd and $q \equiv 0 \pmod 3$.

j	$\mathbf{v}$	j	$\mathbf{v}$	j	$\mathbf{v}$	j	$\mathbf{v}$
1	$(1,19,21,24)$	2	$(1,19,22)$	3	$(1,20,21)$	4	$(4,6,7,19,24)$
5	$(4,6,7,20)$	6	$(5,6,7,21,24)$	7	$(5,6,7,22)$	8	(10)

Table (11.10.4-A). CHEVIE sets $\mathbf{X}_h$, their degrees and parameters for $G_2(q)$ with q odd and $q \equiv 1 \pmod 3$.

h	$\deg \chi_h$	h	$\deg \chi_h$
1	1	2	$q(q+1)^2(q^2+q+1)/6$
3	$q(q+1)^2(q^2-q+1)/2$	4,5	$q(q^2+q+1)(q^2-q+1)/3$
6	$q(q-1)^2(q^2-q+1)/6$	7	$q(q-1)^2(q^2+q+1)/2$
8,9	$q(q-1)^2(q+1)^2/3$	10	q^6
11	$(q^2+q+1)(q^2-q+1)$	12,13	$q(q^2+q+1)(q^2-q+1)$
14	$q^2(q^2+q+1)(q^2-q+1)$	15	$(q+1)(q^2-q+1)$
16	$q(q+1)^2(q^2-q+1)$	17	$q^3(q+1)(q^2-q+1)$
18,20	$(q^2+q+1)(q+1)(q^2-q+1)$	19,21	$q(q^2+q+1)(q+1)(q^2-q+1)$
22,24	$(q^2-q+1)(q-1)(q^2+q+1)$	23,25	$q(q^2-q+1)(q-1)(q^2+q+1)$
26	$(q+1)^2(q^2+q+1)(q^2-q+1)$	27	$(q-1)^2(q^2-q+1)(q^2+q+1)$
28,29	$(q-1)(q+1)(q^2+q+1)(q^2-q+1)$	30	$(q-1)^2(q+1)^2(q^2-q+1)$
31	$(q+1)^2(q-1)^2(q^2+q+1)$		

h	I_h	exceptions
$1,\dots,17$	$-$	
18,19	$k \in \mathbb{Z}_{q-1}$	$(q-1)/3, (q-1)/2 \mid k$
20,21	$k \in \mathbb{Z}_{q-1}$	$(q-1)/2 \mid k$
$22,\dots,25$	$k \in \mathbb{Z}_{q+1}$	$(q+1)/2 \mid k$
26	$(k,l) \in \mathbb{Z}_{q-1}^2$	$(q-1) \mid k, l, (k \pm l), (2k-l), (k-2l)$
27	$(k,l) \in \mathbb{Z}_{q+1}^2$	$(q+1) \mid k, l, (k \pm l), (2k-l), (k-2l)$
28,29	$k \in \mathbb{Z}_{q^2-1}$	$(q \pm 1) \mid k$
30	$k \in \mathbb{Z}_{q^2+q+1}$	$(q^2+q+1)/3 \mid k$
31	$k \in \mathbb{Z}_{q^2-q+1}$	$(q^2-q+1) \mid k$

Table (11.10.4-B). Some Syl_p-decompositions of irreducible characters for $G_2(q)$ with q odd and $q \equiv 1 \pmod 3$.

h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$
8	(9)	12	$(11,22)$	13	$(11,24)$	14	$(12,25)$	14	$(13,23)$
17	$(15,28)$	18	$(11,13)$	18	$(15,16)$	19	$(12,14)$	19	$(16,17)$
19	$(18,28)$	20	$(11,12)$	21	$(13,14)$	21	$(20,29)$	23	$(22,27)$
25	$(24,27)$	26	$(18,19)$	26	$(20,21)$	28	$(22,23)$	29	$(24,25)$
16	$(2,3,9)$	10	$(1,22,24,27)$						

Table (11.10.4-C). Syl_p-vanishing characters of degree $St(1)$ for $G_2(q)$ with q odd and $q \equiv 1 \pmod 3$.

j	$\mathbf{v}$	j	$\mathbf{v}$	j	$\mathbf{v}$	j	$\mathbf{v}$
1	$(1,22,24,27)$	2	$(1,22,25)$	3	$(1,23,24)$	4	$(4,6,7,22,27)$
5	$(4,6,7,23)$	6	$(5,6,7,24,27)$	7	$(5,6,7,25)$	8	(10)

Table (11.10.5-A). CHEVIE sets $\mathbf{X}_h$, their degrees and parameters for $G_2(q)$ with q odd and $q \equiv -1 \pmod 3$.

h	$\deg \chi_h$	h	$\deg \chi_h$
1	1	2	$q(q+1)^2(q^2+q+1)/6$
3	$q(q+1)^2(q^2-q+1)/2$	4,5	$q(q^2+q+1)(q^2-q+1)/3$
6	$q(q-1)^2(q^2-q+1)/6$	7	$q(q-1)^2(q^2+q+1)/2$
8,9	$q(q-1)^2(q+1)^2/3$	10	q^6
11	$(q^2+q+1)(q^2-q+1)$	12,13	$q(q^2+q+1)(q^2-q+1)$
14	$q^2(q^2+q+1)(q^2-q+1)$	15	$(q-1)(q^2+q+1)$
16	$q(q-1)^2(q^2+q+1)$	17	$q^3(q-1)(q^2+q+1)$
18,20	$(q^2+q+1)(q+1)(q^2-q+1)$	19,21	$q(q^2+q+1)(q+1)(q^2-q+1)$
22,24	$(q^2-q+1)(q-1)(q^2+q+1)$	23,25	$q(q^2-q+1)(q-1)(q^2+q+1)$
26	$(q+1)^2(q^2+q+1)(q^2-q+1)$	27	$(q-1)^2(q^2-q+1)(q^2+q+1)$
28,29	$(q-1)(q+1)(q^2+q+1)(q^2-q+1)$	30	$(q-1)^2(q+1)^2(q^2-q+1)$
31	$(q+1)^2(q-1)^2(q^2+q+1)$		

h	I_h	exceptions
$1,\dots,17$	$-$	
$18,\dots,21$	$k \in \mathbb{Z}_{q-1}$	$(q-1)/2 \mid k$
$22,23$	$k \in \mathbb{Z}_{q+1}$	$(q+1)/2 \mid k$
$24,25$	$k \in \mathbb{Z}_{q+1}$	$(q+1)/2, (q+1)/3 \mid k$
26	$(k,l) \in \mathbb{Z}^2_{q-1}$	$(q-1) \mid k, l, (k \pm l), (2k-l), (k-2l)$
27	$(k,l) \in \mathbb{Z}^2_{q+1}$	$(q+1) \mid k, l, (k \pm l), (2k-l), (k-2l)$
$28,29$	$k \in \mathbb{Z}_{q^2-1}$	$(q \pm 1) \mid k$
30	$k \in \mathbb{Z}_{q^2+q+1}$	$(q^2+q+1) \mid k$
31	$k \in \mathbb{Z}_{q^2-q+1}$	$(q^2-q+1)/3 \mid k$

Table (11.10.5-B). Some Syl_p-decompositions of irreducible characters for $G_2(q)$ with q odd and $q \equiv -1 \pmod 3$.

h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$
8	(9)	12	$(11,22)$	13	$(11,24)$	14	$(12,25)$	14	$(13,23)$
17	$(16,25)$	18	$(11,13)$	19	$(12,14)$	19	$(18,28)$	20	$(11,12)$
21	$(13,14)$	21	$(20,29)$	23	$(22,27)$	24	$(15,16)$	25	$(24,27)$
26	$(18,19)$	26	$(20,21)$	28	$(22,23)$	29	$(15,17)$	29	$(24,25)$

h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$
4	$(1,9,15)$	16	$(6,7,9)$	10	$(1,6,7,9,15,22,27)$	31	$(6,6,6,7,7,7,9,9,9,27)$

Table (11.10.5-C). Syl_p-vanishing characters of degree $St(1)$ for $G_2(q)$ with q odd and $q \equiv -1 \pmod 3$.

j	$\mathbf{v}$	j	$\mathbf{v}$	j	$\mathbf{v}$	j	$\mathbf{v}$
1	$(5,6,6,7,7,9,15,27)$	2	$(1,6,7,9,15,22,27)$	3	$(1,6,7,8,15,22,27)$	4	$(1,6,7,8,15,23)$
5	$(1,6,7,9,15,23)$	6	$(1,15,16,22,27)$	7	$(1,15,16,23)$	8	$(1,22,24,27)$
9	$(1,22,25)$	10	$(1,23,24)$	11	$(4,6,7,22,27)$	12	$(4,6,7,23)$
13	$(5,6,6,7,7,8,15,27)$	14	$(5,6,7,15,16,27)$	15	$(5,6,7,24,27)$	16	$(5,6,7,25)$
17	(10)						

11.11. Groups $^2F_4(q^2)$.

Table (11.11-A). Some CHEVIE sets $\mathbf{X}_h$, their degrees and parameters for $^2F_4(q^2)$.

h	$\deg\chi_h$
$2,3$	$\frac{q\sqrt{2}}{2}(q+1)(q^4-q^2+1)(q-1)(q^2+1)^2$
4	$q^2(q^4-q^2+1)(q^8-q^4+1)$
10	$\frac{q^4}{6}(q-1)^2(q+1)^2(q^2+1)^2(q^8-q^4+1)$
$11,12$	$\frac{q^4}{4}(q^4-q^2+1)(q^4-\sqrt{2}q^3+q^2-\sqrt{2}q+1)(q^2+1)^2(q+1)^2(q-1)^2$
$13,14$	$\frac{q^4}{4}(q^4-q^2+1)(q^4+\sqrt{2}q^3+q^2+\sqrt{2}q+1)(q^2+1)^2(q+1)^2(q-1)^2$
17	$\frac{q^7}{3}(q-1)^2(q+1)^2(q^4-q^2+1)(q^8-q^4+1)$
$19,20$	$\frac{q^{13}\sqrt{2}}{2}(q+1)(q^4-q^2+1)(q-1)(q^2+1)^2$
21	q^{24}
46	$(q+1)(q^4-q^2+1)(q^2-\sqrt{2}q+1)(q-1)(q^8-q^4+1)(q^2+1)^2$
$47,48$	$\frac{q\sqrt{2}}{2}(q^4-q^2+1)(q^2-\sqrt{2}q+1)(q^8-q^4+1)(q^2+1)^2(q+1)^2(q-1)^2$
49	$q^4(q+1)(q^4-q^2+1)(q^2-\sqrt{2}q+1)(q-1)(q^8-q^4+1)(q^2+1)^2$
51	$(q^4-q^2+1)(q^8-q^4+1)(q^2+1)^2(q+1)^2(q^2-\sqrt{2}q+1)^2(q-1)^2$

h	I_h	exceptions
$2,3,4,10,11,12,13,14,17,19,20,21$	$-$	
$46,\dots,49$	$k\in\mathbb{Z}_{q^2+q\sqrt{2}+1}$	$(q^2+q\sqrt{2}+1)\mid k$
51	$(k,l)\in(\mathbb{Z}_{q^2+q\sqrt{2}+1})^2$	$q^2+q\sqrt{2}+1\mid k,l,(k\pm l),(l\pm q^2k)$

Table (11.11-B). Some Syl_p-decompositions of irreducible characters for $^2F_4(q^2)$.

h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$
15	(16)	23	$(2,19)$	24	$(3,20)$	27	$(26,34)$
28	$(22,25)$	28	$(26,27)$	31	$(30,33)$	32	$(29,30)$
33	$(32,38)$	34	$(29,31)$	34	$(32,33)$	35	$(42,45)$
36	$(46,49)$	39	$(30,30,30,38)$	45	$(37,42,43,44)$	49	$(46,47,48,51)$
22	$(1,4,5,6,7)$	25	$(5,6,7,18,21)$	26	$(1,5,6,7,18)$	37	$(47,47,48,48,51)$
50	$(37,43,43,44,44)$						

Table (11.11-C). Syl_p-vanishing characters of degree $St(1)$ for $^2F_4(q^2)$.

j	$\mathbf{v}$	j	$\mathbf{v}$
1	$(2,4,10,11,13,17,20,46,47,47,48,51)$	2	$(3,4,10,12,14,17,19,46,47,48,48,51)$
3	$(2,4,10,11,13,17,20,47,49)$	4	$(3,4,10,12,14,17,19,48,49)$
5	(21)		

11.12. Groups $GL(4,q)$.

Table (11.12-A). CHEVIE sets $\mathbf{X}_h$, their degrees and parameters for $GL(4,q)$.

h	$\deg\chi_h$	h	$\deg\chi_h$
1	q^6	2	$q^3(q^2+q+1)$
3	$q^2(q^2+1)$	4	$q(q^2+q+1)$
5	1	6	$(q+1)(q^2+1)q^3$
7	$q(q^2+1)(q+1)^2$	8	$(q^2+1)(q+1)$
9	$q^2(q^2+1)(q^2+q+1)$	10	$q(q^2+1)(q^2+q+1)$
11	$q(q^2+1)(q^2+q+1)$	12	$(q^2+1)(q^2+q+1)$
13	$q^2(q^2+q+1)(q-1)^2$	14	$(q^2+q+1)(q-1)^2$
15	$q(q+1)(q^2+q+1)(q^2+1)$	16	$(q+1)(q^2+1)(q^2+q+1)$
17	$q(q-1)(q^2+q+1)(q^2+1)$	18	$(q-1)(q^2+q+1)(q^2+1)$
19	$(q^2+1)(q^2+q+1)(q+1)^2$	20	$(q-1)(q+1)(q^2+q+1)(q^2+1)$
21	$(q^2+1)(q^2+q+1)(q-1)^2$	22	$(q^2+1)(q-1)^2(q+1)^2$
23	$(q+1)(q^2+q+1)(q-1)^3$		

h	I_h	exceptions
$1\ldots,5$	$k_1 \in \mathbb{Z}_{q-1}$	
$6,\ldots,12$	$(k_1,k_2) \in \mathbb{Z}_{q-1} \times \mathbb{Z}_{q-1}$	$(q-1) \mid (k_1-k_2)$
$13,14$	$k_1 \in \mathbb{Z}_{q^2-1}$	$(q+1) \mid k_1$
$15,16$	$(k_1,k_2,k_3) \in \mathbb{Z}_{q-1}^3$	$(q-1) \mid (k_i-k_j), i \neq j$
$17,18$	$(k_1,k_2) \in \mathbb{Z}_{q-1} \times \mathbb{Z}_{q^2-1}$	$(q^2-1) \mid (q+1)k_1 - k_2$
		$(q+1) \mid k_2$
19	$(k_1,k_2,k_3,k_4) \in \mathbb{Z}_{q-1}^4$	$(q-1) \mid (k_i-k_j), i \neq j$
20	$(k_1,k_2,k_3) \in \mathbb{Z}_{q^2-1} \times \mathbb{Z}_{q-1} \times \mathbb{Z}_{q-1}$	$(q+1) \mid k_1$
		$(q-1) \mid (k_2-k_3)$
21	$(k_1,k_2) \in \mathbb{Z}_{q^2-1} \times \mathbb{Z}_{q^2-1}$	$(q+1) \mid k_i$
		$(q^2-1) \mid qk_1 - k_2$
		$(q^2-1) \mid (k_1-k_2)$
22	$(k_1,k_2) \in \mathbb{Z}_{q^3-1} \times \mathbb{Z}_{q-1}$	$(q^2+q+1) \mid (q+1)k_1$
23	$k_1 \in \mathbb{Z}_{q^4-1}$	$(q^3+q+q^2+1) \mid (q^2+q+1)k_1$
		$(q^2+1) \mid k_1$

Table (11.12-B). Some Syl_p-decompositions of irreducible characters for $GL(4,q)$.

h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$
10	(11)	6	$(1,2)$	6	$(8,20)$	7	$(3,11)$	8	$(4,5)$
9	$(3,6)$	9	$(11,17)$	9	$(12,20)$	11	$(2,4)$	11	$(12,18)$
12	$(3,8)$	13	$(14,23)$	15	$(6,7)$	15	$(9,11)$	15	$(16,20)$
16	$(7,8)$	16	$(11,12)$	17	$(18,21)$	19	$(15,16)$	20	$(17,18)$
21	$(13,14)$	2	$(3,5,18)$	2	$(4,14,18)$	1	$(5,14,18,23)$		

Table (11.12-C). Syl_p-vanishing characters of degree $St(1)$ for $GL(4,q)$.

j	$\mathbf{v}$	j	$\mathbf{v}$	j	$\mathbf{v}$
1	$(5,14,18,23)$	2	$(5,13,18)$	3	(1)

Table (11.12-D). p-vanishing characters of degree $St(1)$ and their values for $GL(4,q)$ with $3 \mid (q+1)$.

In view of Lemma 9.1 we have the character $\psi = 1_G + \chi_{14}(a) + \chi_{18}(0,a) + \chi_{23}((q^2-1)b)$. This character differs from St only on the following classes:

C_h	St	ψ
$C_{13}(i_1)$	q^2	$3q^2 + 2Re(\xi_1^{i_1 a}) - 2(q^2+1)Re(\xi_1^{i_1 a})$
$C_{17}(i_1,i_2)$	$-q$	$q - 2(q-1)Re(\xi_1^{i_2 a}) - 2Re(\xi_1^{i_2 a})$
$C_{20}(i_1,i_2,i_3)$	-1	$1 - 2Re(\xi_1^{i_1 a})$
$C_{21}(i_1,i_2)$	1	$1 + 2(Re(\xi_1^{(i_1+i_2)a}) + Re(\xi_1^{(i_1-i_2)a}) - Re(\xi_1^{i_1 a}) - Re(\xi_1^{i_2 a}))$
$C_{23}(i_1)$	-1	$1 + 2(Re(\xi^{i_1 a}) - Re(\xi_2^{i_1 b}) - Re(\xi_2^{qi_1 b}))$

11.13. Groups $GL(5, q)$.

Table (11.13-A). CHEVIE sets $\mathbf{X}_h$, their degrees and parameters for $GL(5, q)$.

h	$\deg \chi_h$
1	q^{10}
2	$q^6(q + 1)(q^2 + 1)$
3	$(q^4 + q^3 + q^2 + q + 1)q^4$
4	$q^3(q^2 + 1)(q^2 + q + 1)$
5	$q^2(q^4 + q^3 + q^2 + q + 1)$
6	$q(q + 1)(q^2 + 1)$
7	1
8	$q^6(q^4 + q^3 + q^2 + q + 1)$
9	$q^3(q^2 + q + 1)(q^4 + q^3 + q^2 + q + 1)$
10	$q^2(q^2 + 1)(q^4 + q^3 + q^2 + q + 1)$
11	$q(q^2 + q + 1)(q^4 + q^3 + q^2 + q + 1)$
12	$q^4 + q^3 + q^2 + q + 1$
13	$q^4(q^2 + 1)(q^4 + q^3 + q^2 + q + 1)$
14	$q^3(q^2 + 1)(q^4 + q^3 + q^2 + q + 1)$
15	$q^2(q + 1)(q^2 + 1)(q^4 + q^3 + q^2 + q + 1)$
16	$q(q + 1)(q^2 + 1)(q^4 + q^3 + q^2 + q + 1)$
17	$q(q^2 + 1)(q^4 + q^3 + q^2 + q + 1)$
18	$(q^2 + 1)(q^4 + q^3 + q^2 + q + 1)$
19	$q^3(q + 1)(q^2 + 1)(q^4 + q^3 + q^2 + q + 1)$
20	$q(q^2 + 1)(q^4 + q^3 + q^2 + q + 1)(q + 1)^2$
21	$(q + 1)(q^2 + 1)(q^4 + q^3 + q^2 + q + 1)$
22	$q^3(q - 1)(q^2 + 1)(q^4 + q^3 + q^2 + q + 1)$
23	$q(q - 1)(q + 1)(q^2 + 1)(q^4 + q^3 + q^2 + q + 1)$
24	$(q - 1)(q^2 + 1)(q^4 + q^3 + q^2 + q + 1)$
25	$q^2(q^2 + q + 1)(q^4 + q^3 + q^2 + q + 1)(q^2 + 1)$
26	$q(q^2 + q + 1)(q^4 + q^3 + q^2 + q + 1)(q^2 + 1)$
27	$q(q^2 + q + 1)(q^4 + q^3 + q^2 + q + 1)(q^2 + 1)$
28	$(q^2 + q + 1)(q^4 + q^3 + q^2 + q + 1)(q^2 + 1)$
29	$q^2(q^2 + q + 1)(q^4 + q^3 + q^2 + q + 1)(q - 1)^2$
30	$(q^2 + q + 1)(q^4 + q^3 + q^2 + q + 1)(q - 1)^2$
31	$q(q + 1)(q^2 + q + 1)(q^4 + q^3 + q^2 + q + 1)(q^2 + 1)$
32	$(q + 1)(q^2 + q + 1)(q^4 + q^3 + q^2 + q + 1)(q^2 + 1)$
33	$q(q - 1)(q^2 + q + 1)(q^4 + q^3 + q^2 + q + 1)(q^2 + 1)$
34	$(q - 1)(q^2 + q + 1)(q^4 + q^3 + q^2 + q + 1)(q^2 + 1)$
35	$q(q + 1)(q^2 + 1)(q^4 + q^3 + q^2 + q + 1)(q - 1)^2$
36	$(q + 1)(q^2 + 1)(q^4 + q^3 + q^2 + q + 1)(q - 1)^2$
37	$(q^2 + q + 1)(q^4 + q^3 + q^2 + q + 1)(q^2 + 1)(q + 1)^2$
38	$(q - 1)(q + 1)(q^2 + q + 1)(q^4 + q^3 + q^2 + q + 1)(q^2 + 1)$
39	$(q^2 + q + 1)(q^4 + q^3 + q^2 + q + 1)(q^2 + 1)(q - 1)^2$
40	$(q^2 + 1)(q^4 + q^3 + q^2 + q + 1)(q - 1)^2(q + 1)^2$
41	$(q + 1)(q^2 + 1)(q^4 + q^3 + q^2 + q + 1)(q - 1)^3$
42	$(q + 1)(q^2 + q + 1)(q^4 + q^3 + q^2 + q + 1)(q - 1)^3$
43	$(q^2 + 1)(q^2 + q + 1)(q + 1)^2(q - 1)^4$

h	I_h	exceptions
$1\ldots,7$	$k_1 \in \mathbb{Z}_{q-1}$	
$8,\ldots,18$	$(k_1,k_2) \in \mathbb{Z}_{q-1} \times \mathbb{Z}_{q-1}$	$(q-1) \mid (k_1 - k_2)$
$19,20,21$	$(k_1,k_2,k_3) \in \mathbb{Z}_{q-1}^3$	$(q-1) \mid (k_i - k_j), i \neq j$
$22,23,24$	$(k_1,k_2) \in \mathbb{Z}_{q-1} \times \mathbb{Z}_{q^2-1}$	$(q^2-1) \mid (q+1)k_1 - k_2$
		$(q+1) \mid k_2$
$25,\ldots,28$	$(k_1,k_2,k_3) \in \mathbb{Z}_{q-1}^3$	$(q-1) \mid (k_i - k_j), i \neq j$
$29,30$	$(k_1,k_2) \in \mathbb{Z}_{q^2-1} \times \mathbb{Z}_{q-1}$	$(q+1) \mid k_1$
$31,32$	$(k_1,k_2,k_3,k_4) \in \mathbb{Z}_{q-1}^4$	$(q-1) \mid (k_i - k_j), i \neq j$
$33,34$	$(k_1,k_2,k_3) \in \mathbb{Z}_{q-1} \times \mathbb{Z}_{q^2-1} \times \mathbb{Z}_{q-1}$	$(q-1) \mid (k_1 - k_3)$
		$(q^2-1) \mid (q+1)k_3 - k_2$
		$(q^2-1) \mid (q+1)k_1 - qk_2$
		$(q+1) \mid k_2$
$35,36$	$(k_1,k_2) \in \mathbb{Z}_{q-1} \times \mathbb{Z}_{q^3-1}$	$(q^3-1) \mid (q^2+q+1)k_1 - k_2$
		$(q^2+q+1) \mid (q+1)k_2$
37	$(k_1,k_2,k_3,k_4,k_5) \in \mathbb{Z}_{q-1}^5$	$(q-1) \mid k_i - k_j, i \neq j$
38	$(k_1,k_2,k_3,k_4) \in \mathbb{Z}_{q^2-1} \times \mathbb{Z}_{q-1}^3$	$(q+1) \mid k_1$
		$(q-1) \mid k_i - k_j, 2 \leq i < j \leq 4$
39	$(k_1,k_2,k_3) \in \mathbb{Z}_{q^2-1} \times \mathbb{Z}_{q-1} \times \mathbb{Z}_{q^2-1}$	$(q+1) \mid k_1, k_3$
		$(q^2-1) \mid qk_1 - k_3$
		$(q^2-1) \mid k_1 - k_3$
40	$(k_1,k_2,k_3) \in \mathbb{Z}_{q^3-1} \times \mathbb{Z}_{q-1} \times \mathbb{Z}_{q-1}$	$(q^2+q+1) \mid (q+1)k_1$
		$(q-1) \mid k_2 - k_3$
41	$(k_1,k_2) \in \mathbb{Z}_{q^3-1} \times \mathbb{Z}_{q^2-1}$	$(q^2+q+1) \mid (q+1)k_1$
		$(q+1) \mid k_2$
42	$(k_1,k_2) \in \mathbb{Z}_{q^4-1} \times \mathbb{Z}_{q-1}$	$(q^3+q^2+q+1) \mid (q^2+q+1)k_1$
		$(q^2+1) \mid k_1$
43	$k_1 \in \mathbb{Z}_{q^5-1}$	$(q^5-1)/(q-1) \mid (q^4-1)/(q-1)k_1$

Table (11.13-B). Some Syl_p-decompositions of irreducible characters for $GL(5,q)$.

h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$
26	(27)	2	$(6,23)$	8	$(1,2)$	9	$(3,14)$	10	$(3,5)$
11	$(5,17)$	12	$(6,7)$	13	$(3,8)$	13	$(14,22)$	13	$(17,33)$
14	$(2,4)$	14	$(17,23)$	14	$(18,34)$	15	$(5,9)$	15	$(10,14)$
15	$(16,23)$	16	$(3,11)$	16	$(10,17)$	17	$(4,6)$	17	$(18,24)$
18	$(5,12)$	19	$(8,9)$	19	$(13,14)$	19	$(21,38)$	20	$(10,27)$
20	$(15,16)$	21	$(11,12)$	21	$(17,18)$	22	$(24,39)$	25	$(10,19)$
25	$(13,15)$	25	$(27,33)$	25	$(28,38)$	27	$(9,11)$	27	$(14,16)$
27	$(15,17)$	27	$(28,34)$	28	$(10,21)$	28	$(16,18)$	29	$(30,42)$
31	$(19,20)$	31	$(25,27)$	31	$(32,38)$	32	$(20,21)$	32	$(27,28)$
33	$(22,23)$	33	$(34,39)$	34	$(23,24)$	35	$(36,41)$	37	$(31,32)$
38	$(33,34)$	39	$(29,30)$	40	$(35,36)$	3	$(5,24,30)$	4	$(5,7,24)$
1	$(7,24,30,42)$								

Table (11.13-C). Syl_p-vanishing characters of degree $St(1)$ for $GL(5,q)$.

j	$\mathbf{v}$	j	$\mathbf{v}$	j	$\mathbf{v}$
3	$(7,24,30,42)$	2	$(7,24,29)$	1	(1)

Table (11.13-D). p-vanishing characters of degree $St(1)$ and their values for $GL(5, q)$ with $3 \mid (q + 1)$.

In view of Lemma 9.2 we have the character $\psi = 1_G + \chi_{24}(0, a) + \chi_{30}(a, 0) + \chi_{42}((q^2 - 1)b, 0)$. This character differs from St only on the following classes:

C_h	St	ψ
$C_{22}(i_1, i_2)$	$-q^3$	$q^3 - 2q^3 Re(\xi_1^{a i_2})$
$C_{29}(i_1, i_2)$	q^2	$3q^2 - 2q^2 Re(\xi_1^{a i_1})$
$C_{33}(i_1, i_2, i_3)$	$-q$	$q - 2q Re(\xi_1^{a i_2})$
$C_{38}(i_1, i_2, i_3, i_4)$	-1	$1 - 2Re(\xi_1^{a i_1})$
$C_{39}(i_1, i_2, i_3)$	1	$1 - 2(Re(\xi_1^{a i_1}) + Re(\xi_1^{a i_3}) - Re(\xi_1^{(i_1 + i_3)a}) - Re(\xi_1^{(i_1 - i_3)a}))$
$C_{41}(i_1, i_2)$	-1	$1 - 2Re(\xi_1^{a i_2})$
$C_{42}(i_1, i_2)$	-1	$1 + 2Re(\xi_1^{a i_1}) - 2Re(\xi_2^{i_1 b}) - 2Re(\xi_2^{q i_1 b})$

11.14. **Groups** $GL(6, q)$.

Table (11.14-A). Some CHEVIE sets $\mathbf{X}_h$, their degrees and parameters for $GL(6, q)$.

h	$\deg \chi_h$
6	$q^4(q^2 + 1)(q^2 - q + 1)(q + 1)^3$
7	$q^3(q^2 - q + 1)(q^4 + q^3 + q^2 + q + 1)$
9	$q^2(q^2 - q + 1)(q^2 + q + 1)^2$
10	$q(q^4 + q^3 + q^2 + q + 1)$
11	1
39	$q^2(q^2 + q + 1)(q^4 + q^3 + q^2 + q + 1)(q^2 + 1)(q - 1)^3$
40	$(q^2 + q + 1)(q^4 + q^3 + q^2 + q + 1)(q - 1)^3$
48	$q^2(q - 1)(q^2 - q + 1)(q^2 + q + 1)(q^4 + q^3 + q^2 + q + 1)(q^2 + 1)$
49	$q(q - 1)(q^2 - q + 1)(q^4 + q^3 + q^2 + q + 1)(q^2 + q + 1)^2$
50	$(q - 1)(q^2 - q + 1)(q^2 + q + 1)(q^4 + q^3 + q^2 + q + 1)$
68	$(q^2 - q + 1)(q^4 + q^3 + q^2 + q + 1)(q - 1)^2(q^2 + q + 1)^2$
70	$(q^2 + 1)(q^4 + q^3 + q^2 + q + 1)(q + 1)^2(q - 1)^4$
78	$q(q^2 - q + 1)(q^4 + q^3 + q^2 + q + 1)(q^2 + 1)(q - 1)^2(q + 1)^3$
79	$(q^2 - q + 1)(q^4 + q^3 + q^2 + q + 1)(q^2 + 1)(q - 1)^2(q + 1)^2$
99	$(q + 1)(q^2 - q + 1)(q^4 + q^3 + q^2 + q + 1)(q^2 + q + 1)^2(q - 1)^3$
107	$(q^2 + q + 1)(q^2 - q + 1)(q^4 + q^3 + q^2 + q + 1)(q^2 + 1)(q + 1)^2(q - 1)^3$
110	$(q + 1)(q^2 - q + 1)(q^4 + q^3 + q^2 + q + 1)(q^2 + q + 1)^2(q - 1)^4$
111	$(q^2 - q + 1)(q^2 + 1)(q^2 + q + 1)^2(q + 1)^3(q - 1)^4$
112	$(q^2 + q + 1)(q^4 + q^3 + q^2 + q + 1)(q^2 + 1)(q + 1)^2(q - 1)^5$

h	I_h	exceptions
$6, 7, 9, 10, 11$	$k_1 \in \mathbb{Z}_{q-1}$	
$39, 40$	$k_1 \in \mathbb{Z}_{q^2-1}$	$(q + 1) \mid k_1$
$48, 49, 50, 68$	$(k_1, k_2) \in \mathbb{Z}_{q-1} \times \mathbb{Z}_{q^2-1}$	$(q^2 - 1) \mid (q + 1)k_1 - k_2$
		$(q + 1) \mid k_2$
70	$k_1 \in \mathbb{Z}_{q^3-1}$	$(q^2 + q + 1) \mid (q + 1)k_1$
$78, 79$	$(k_1, k_2) \in \mathbb{Z}_{q-1} \times \mathbb{Z}_{q^3-1}$	$(q^3 - 1) \mid (q^2 + q + 1)k_1 - k_2$
		$(q^2 + q + 1) \mid (q + 1)k_2$
99	$(k_1, k_2) \in \mathbb{Z}_{q-1} \times \mathbb{Z}_{q^4-1}$	$(q^4 - 1) \mid (q^3 + q^2 + q + 1)k_1 - k_2$
		$(q^3 + q^2 + q + 1) \mid (q^2 + q + 1)k_2$
		$(q^2 + 1) \mid k_2$
107	$(k_1, k_2, k_3) \in \mathbb{Z}_{q^3-1} \times \mathbb{Z}_{q^2-1} \times \mathbb{Z}_{q-1}$	$(q^2 + q + 1) \mid (q + 1)k_1$
		$(q + 1) \mid k_2$
110	$(k_1, k_2) \in \mathbb{Z}_{q^4-1} \times \mathbb{Z}_{q^2-1}$	$(q^3 + q^2 + q + 1) \mid (q^2 + q + 1)k_1$
		$(q^4 - 1) \mid qk_1 - (q^2 + 1)k_2$
		$(q + 1) \mid k_2, (q^2 + 1) \mid k_1$
111	$(k_1, k_2) \in \mathbb{Z}_{q^5-1} \times \mathbf{Z}_{q-1}$	$(q^5 - 1)/(q - 1) \mid (q^4 - 1)k_1/(q - 1)$
112	$k_1 \in \mathbb{Z}_{q^6-1}$	$(q^6 - 1)/(q - 1) \mid (q^5 - 1)k_1/(q - 1)$
		$(q^4 + q^2 + 1) \mid (q^2 + 1)k_1$
		$(q^3 + 1) \mid k_1$

Table (11.14-B). Some Syl_p-decompositions of irreducible characters for $GL(6,q)$.

h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$
30	(32)	31	(35)	34	(36)	58	(59)	58	(61)
59	(61)	60	(62)	60	(63)	62	(63)	81	(82)
85	(86)	12	(1, 2)	13	(3, 20)	13	(17, 75)	15	(6, 35)
17	(9, 27)	18	(10, 11)	19	(3, 12)	19	(20, 46)	20	(2, 5)
21	(4, 32)	21	(14, 20)	21	(22, 47)	22	(3, 15)	23	(24, 48)
25	(9, 15)	25	(26, 49)	26	(7, 36)	26	(16, 27)	27	(8, 10)
27	(28, 50)	28	(9, 18)	29	(4, 19)	29	(35, 74)	32	(6, 13)
32	(36, 75)	33	(35, 43)	35	(5, 8)	35	(37, 76)	36	(6, 17)
37	(7, 28)	38	(40, 110)	41	(12, 13)	41	(19, 20)	42	(14, 52)
42	(21, 22)	43	(14, 16)	43	(23, 24)	44	(16, 55)	44	(25, 26)
45	(17, 18)	45	(27, 28)	51	(14, 41)	51	(19, 21)	51	(29, 32)
51	(52, 74)	51	(55, 94)	52	(13, 15)	52	(20, 22)	52	(32, 35)
52	(55, 75)	52	(56, 95)	53	(16, 42)	53	(23, 61)	53	(32, 33)
53	(43, 52)	53	(54, 75)	54	(14, 44)	54	(24, 63)	54	(33, 36)
54	(43, 55)	55	(15, 17)	55	(25, 27)	55	(35, 36)	55	(56, 76)
56	(16, 45)	56	(26, 28)	56	(36, 37)	57	(23, 51)	57	(61, 84)
57	(63, 94)	61	(21, 25)	61	(24, 52)	61	(63, 86)	61	(64, 95)
63	(22, 26)	63	(23, 55)	63	(64, 87)	64	(24, 56)	65	(66, 98)
65	(67, 90)	66	(68, 91)	67	(68, 99)	69	(70, 112)	71	(41, 42)
71	(51, 52)	71	(73, 103)	72	(43, 82)	72	(53, 54)	73	(44, 45)
73	(55, 56)	74	(46, 47)	74	(76, 104)	75	(48, 86)	76	(49, 50)
77	(79, 107)	80	(43, 71)	80	(51, 53)	80	(57, 61)	80	(82, 94)
80	(83, 103)	82	(42, 44)	82	(52, 54)	82	(53, 55)	82	(61, 63)
82	(83, 95)	83	(43, 73)	83	(54, 56)	83	(63, 64)	84	(48, 74)
84	(86, 100)	84	(87, 104)	86	(47, 49)	86	(87, 101)	87	(48, 76)
88	(65, 67)	88	(89, 109)	89	(66, 68)	90	(38, 39)	90	(91, 110)
91	(39, 40)	92	(71, 72)	92	(80, 82)	92	(93, 103)	93	(72, 73)
93	(82, 83)	94	(74, 75)	94	(84, 86)	94	(95, 104)	95	(75, 76)
95	(86, 87)	96	(77, 78)	96	(97, 107)	97	(78, 79)	98	(99, 110)
100	(65, 66)	100	(101, 105)	101	(67, 68)	102	(92, 93)	103	(94, 95)
104	(88, 89)	104	(100, 101)	105	(90, 91)	106	(96, 97)	108	(69, 70)
109	(98, 99)	4	(7, 50, 68)	5	(7, 10, 49)	8	(9, 11, 50)	14	(3, 4, 6)

h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$
16	(6, 7, 9)	24	(4, 6, 9)	24	(16, 50, 68)
3	(9, 48, 50, 68)	2	(10, 39, 49, 68, 99)	47	(48, 50, 68, 68, 99)
1	(11, 40, 50, 68, 99, 110)	46	(39, 40, 50, 68, 68, 99, 110)		

Table (11.14-C). Syl_p-vanishing characters of degree $St(1)$ for $GL(6,q)$.

j	$\mathbf{v}$	j	$\mathbf{v}$	j	$\mathbf{v}$
1	(11, 40, 50, 68, 99, 110)	2	(11, 40, 50, 67, 110)	3	(11, 38, 50, 68, 99)
4	(11, 40, 50, 68, 98)	5	(11, 38, 50, 67)	6	(1)

11.15. Groups $CSp(6,q)$.

Table (11.15-A). Some CHEVIE sets $\mathbf{X}_h$, their degrees and parameters for $CSp(6,q)$ with q odd.

h	$\deg \chi_h$	k	$\deg \chi_h$
1	1	4	$\frac{1}{2}q(q^2 - q + 1)(q + 1)^2$
5	$\frac{1}{2}(q - 1)^2 q(q^2 + q + 1)$	7	$q^3(q^2 - q + 1)(q^2 + q + 1)$
10	$\frac{1}{2}q^4(q^2 - q + 1)(q^2 + 1)$	11	$\frac{1}{2}(q - 1)^2 q^4(q^2 + q + 1)$
18	$(q - 1)(q^2 + q + 1)$	45	$\frac{1}{2}(q - 1)^3 q(q^2 + q + 1)(q^2 - q + 1)$
50	$(q - 1)^2(q^2 + q + 1)(q^2 + 1)$	51	$(q - 1)^3 q(q^2 + 1)(q^2 + q + 1)$
65	$(q - 1)(q^2 - q + 1)(q^2 + 1)(q^2 + q + 1)$	71	$(q - 1)^2(q + 1)(q^2 + q + 1)(q^2 - q + 1)$
97	$(q - 1)^2(q + 1)^3(q^2 + 1)(q^2 - q + 1)$	98	$(q - 1)^3(q^2 + q + 1)(q^2 + 1)(q^2 - q + 1)$

h	I_h	exceptions
$1, 4, 5, 7, 10, 11, 18$	$k_1 \in \mathbb{Z}_{q-1}$	
45	$(k_1, k_2) \in \mathbb{Z}_{q-1} \times \mathbb{Z}_{q-1}$	$(q+1) \mid k_1$
$50, 51, 65, 71$	$(k_1, k_2) \in \mathbb{Z}_{q+1} \times \mathbb{Z}_{q-1}$	$\frac{q+1}{2} \mid k_1$
97	$(k_1, k_2) \in \mathbb{Z}_{q^3-1} \times \mathbb{Z}_{q-1}$	$(q^2 + q + 1) \mid k_1$
98	$(k_1, k_2, k_3, k_4) \in \mathbb{Z}_{q+1}^3 \times \mathbb{Z}_{q-1}$	$(q+1) \mid k_1, k_2, k_3$
		$(q+1) \mid (k_i \pm k_j), 1 \le i < j \le 3$

Table (11.15-B). Some Syl_p-decompositions of irreducible characters for $CSp(6, q)$, q odd.

h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$
24	(26)	27	(29)	54	(55)	62	(63)	9	(4, 42)	14	(23, 41)
14	(31, 35)	16	(37, 42)	19	(18, 50)	20	(50, 61)	21	(39, 44)	22	(21, 52)
23	(19, 35)	25	(23, 61)	26	(23, 65)	28	(23, 79)	28	(26, 66)	28	(37, 44)
29	(25, 67)	29	(26, 63)	30	(16, 46)	30	(25, 80)	30	(28, 64)	30	(29, 68)
31	(19, 41)	32	(31, 71)	33	(31, 89)	33	(39, 42)	34	(21, 46)	34	(32, 90)
34	(33, 72)	38	(14, 42)	38	(25, 43)	39	(5, 11)	40	(16, 34)	40	(21, 30)
42	(32, 39)	43	(19, 39)	47	(13, 14)	48	(15, 55)	49	(16, 17)	49	(47, 93)
52	(51, 86)	53	(15, 47)	53	(23, 25)	55	(14, 16)	55	(26, 29)	55	(36, 39)
55	(37, 38)	55	(53, 83)	56	(15, 49)	56	(28, 30)	56	(53, 93)	56	(55, 84)
57	(23, 26)	58	(25, 29)	58	(57, 77)	59	(26, 28)	59	(57, 79)	60	(29, 30)
63	(61, 85)	64	(20, 52)	64	(61, 95)	64	(63, 86)	66	(65, 87)	67	(65, 85)
68	(66, 86)	68	(67, 88)	69	(31, 32)	70	(33, 34)	70	(69, 96)	72	(71, 99)
73	(47, 48)	73	(53, 55)	73	(57, 58)	74	(48, 49)	74	(55, 56)	74	(59, 60)
74	(73, 93)	75	(57, 59)	76	(58, 60)	76	(75, 92)	77	(61, 63)	78	(63, 64)
78	(77, 95)	79	(65, 66)	80	(67, 68)	80	(79, 95)	81	(31, 33)	82	(32, 34)
82	(81, 94)	83	(65, 67)	84	(66, 68)	84	(83, 94)	85	(50, 51)	86	(85, 98)
88	(87, 98)	90	(89, 99)	91	(73, 74)	91	(75, 76)	92	(69, 70)	92	(77, 78)
92	(81, 82)	93	(79, 80)	93	(83, 84)	94	(71, 72)	94	(87, 88)	95	(50, 52)
95	(85, 86)	96	(89, 90)								

h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$	h	$\mathbf{v}$
2	(1, 4, 18)	3	(1, 5, 18)	8	(5, 7, 44)	12	(10, 11, 46)
13	(1, 1, 18)	17	(10, 10, 46)	35	(1, 2, 4)	35	(4, 4, 13)
37	(4, 7, 10)	41	(5, 5, 18)	44	(41, 45, 71)	61	(41, 41, 71)
87	(45, 45, 71)	100	(51, 51, 51, 98)	89	(5, 5, 11, 11, 71)	99	(45, 45, 45, 45, 98)
6	(1, 4, 5, 18, 18, 50)						

Table (11.15-C). Syl_p-vanishing characters of degree $St(1)$ for $CSp(6, q)$ with q odd.

j	$\mathbf{v}$	j	$\mathbf{v}$
1	$(1, 5, 5, 18, 45, 45, 50, 51, 51, 65, 71, 98)$	2	$(1, 5, 5, 18, 45, 45, 51, 65, 71, 85, 98)$
3	$(1, 5, 5, 18, 45, 45, 51, 65, 71, 86)$	4	$(1, 5, 5, 18, 45, 45, 51, 67, 71, 98)$
5	$(1, 5, 5, 18, 45, 45, 52, 65, 71)$	6	$(1, 5, 5, 18, 50, 51, 51, 65, 87, 98)$
7	$(1, 5, 5, 18, 50, 51, 51, 65, 88)$	8	$(1, 5, 5, 18, 50, 51, 51, 66, 98)$
9	$(1, 5, 5, 18, 51, 65, 85, 87, 98)$	10	$(1, 5, 5, 18, 51, 65, 85, 88)$
11	$(1, 5, 5, 18, 51, 65, 86, 87)$	12	$(1, 5, 5, 18, 51, 66, 85, 98)$
13	$(1, 5, 5, 18, 51, 66, 86)$	14	$(1, 5, 5, 18, 51, 67, 87, 98)$
15	$(1, 5, 5, 18, 51, 67, 88)$	16	$(1, 5, 5, 18, 51, 68)$
17	$(1, 5, 5, 18, 52, 65, 87)$	18	$(1, 5, 5, 18, 52, 66)$
19	$(1, 5, 5, 19, 45, 45, 51, 51, 65, 71, 98)$	20	$(1, 5, 5, 19, 51, 51, 65, 87, 98)$
21	$(1, 5, 5, 19, 51, 51, 65, 88)$	22	$(1, 5, 5, 19, 51, 51, 66, 98)$
23	$(1, 5, 11, 18, 45, 45, 45, 50, 51, 65, 71, 98)$	24	$(1, 5, 11, 18, 45, 45, 45, 65, 71, 85, 98)$
25	$(1, 5, 11, 18, 45, 45, 45, 65, 71, 86)$	26	$(1, 5, 11, 18, 45, 45, 45, 67, 71, 98)$
27	$(1, 5, 11, 18, 45, 50, 51, 65, 87, 98)$	28	$(1, 5, 11, 18, 45, 50, 51, 65, 88)$
29	$(1, 5, 11, 18, 45, 50, 51, 66, 98)$	30	$(1, 5, 11, 18, 45, 65, 85, 87, 98)$

j	$\mathbf{v}$	j	$\mathbf{v}$
31	$(1, 5, 11, 18, 45, 65, 85, 88)$	32	$(1, 5, 11, 18, 45, 65, 86, 87)$
33	$(1, 5, 11, 18, 45, 66, 85, 98)$	34	$(1, 5, 11, 18, 45, 66, 86)$
35	$(1, 5, 11, 18, 45, 67, 87, 98)$	36	$(1, 5, 11, 18, 45, 67, 88)$
37	$(1, 5, 11, 18, 45, 68)$	38	$(1, 5, 11, 19, 45, 45, 45, 51, 65, 71, 98)$
39	$(1, 5, 11, 19, 45, 51, 65, 87, 98)$	40	$(1, 5, 11, 19, 45, 51, 65, 88)$
41	$(1, 5, 11, 19, 45, 51, 66, 98)$	42	$(1, 11, 11, 18, 45, 45, 45, 45, 50, 65, 71, 98)$
43	$(1, 11, 11, 18, 45, 45, 50, 65, 87, 98)$	44	$(1, 11, 11, 18, 45, 45, 50, 65, 88)$
45	$(1, 11, 11, 18, 45, 45, 50, 66, 98)$	46	$(1, 11, 11, 18, 50, 65, 71, 99)$
47	$(1, 11, 11, 18, 50, 65, 72)$	48	$(1, 11, 11, 19, 45, 45, 45, 45, 65, 71, 98)$
49	$(1, 11, 11, 19, 45, 45, 65, 87, 98)$	50	$(1, 11, 11, 19, 45, 45, 65, 88)$
51	$(1, 11, 11, 19, 45, 45, 66, 98)$	52	$(1, 11, 11, 19, 65, 71, 99)$
53	$(1, 11, 11, 19, 65, 72)$	54	$(1, 18, 39, 45, 45, 45, 50, 51, 65, 71, 98)$
55	$(1, 18, 39, 45, 45, 45, 65, 71, 85, 98)$	56	$(1, 18, 39, 45, 45, 45, 65, 71, 86)$
57	$(1, 18, 39, 45, 45, 45, 67, 71, 98)$	58	$(1, 18, 39, 45, 50, 51, 65, 87, 98)$
59	$(1, 18, 39, 45, 50, 51, 65, 88)$	60	$(1, 18, 39, 45, 50, 51, 66, 98)$
61	$(1, 18, 39, 45, 65, 85, 87, 98)$	62	$(1, 18, 39, 45, 65, 85, 88)$
63	$(1, 18, 39, 45, 65, 86, 87)$	64	$(1, 18, 39, 45, 66, 85, 98)$
65	$(1, 18, 39, 45, 66, 86)$	66	$(1, 18, 39, 45, 67, 87, 98)$
67	$(1, 18, 39, 45, 67, 88)$	68	$(1, 18, 39, 45, 68)$
69	$(1, 19, 39, 45, 45, 45, 51, 65, 71, 98)$	70	$(1, 19, 39, 45, 51, 65, 87, 98)$
71	$(1, 19, 39, 45, 51, 65, 88)$	72	$(1, 19, 39, 45, 51, 66, 98)$
73	$(1, 41, 45, 45, 50, 51, 51, 65, 71, 98)$	74	$(1, 41, 45, 45, 51, 65, 71, 85, 98)$
75	$(1, 41, 45, 45, 51, 65, 71, 86)$	76	$(1, 41, 45, 45, 51, 67, 71, 98)$
77	$(1, 41, 45, 45, 52, 65, 71)$	78	$(1, 41, 50, 51, 51, 65, 87, 98)$
79	$(1, 41, 50, 51, 51, 65, 88)$	80	$(1, 41, 50, 51, 51, 66, 98)$
81	$(1, 41, 51, 65, 85, 87, 98)$	82	$(1, 41, 51, 65, 85, 88)$
83	$(1, 41, 51, 65, 86, 87)$	84	$(1, 41, 51, 66, 85, 98)$
85	$(1, 41, 51, 66, 86)$	86	$(1, 41, 51, 67, 87, 98)$
87	$(1, 41, 51, 67, 88)$	88	$(1, 41, 51, 68)$
89	$(1, 41, 52, 65, 87)$	90	$(1, 41, 52, 66)$
91	$(1, 43, 45, 45, 45, 51, 65, 71, 98)$	92	$(1, 43, 45, 51, 65, 87, 98)$
93	$(1, 43, 45, 51, 65, 88)$	94	$(1, 43, 45, 51, 66, 98)$
95	$(1, 44, 45, 50, 51, 51, 65, 98)$	96	$(1, 44, 45, 51, 65, 85, 98)$
97	$(1, 44, 45, 51, 65, 86)$	98	$(1, 44, 45, 51, 67, 98)$
99	$(1, 44, 45, 52, 65)$	100	$(3, 5, 45, 45, 50, 51, 51, 65, 71, 98)$
101	$(3, 5, 45, 45, 51, 65, 71, 85, 98)$	102	$(3, 5, 45, 45, 51, 65, 71, 86)$
103	$(3, 5, 45, 45, 51, 67, 71, 98)$	104	$(3, 5, 45, 45, 52, 65, 71)$
105	$(3, 5, 50, 51, 51, 65, 87, 98)$	106	$(3, 5, 50, 51, 51, 65, 88)$
107	$(3, 5, 50, 51, 51, 66, 98)$	108	$(3, 5, 51, 65, 85, 87, 98)$
109	$(3, 5, 51, 65, 85, 88)$	110	$(3, 5, 51, 65, 86, 87)$
111	$(3, 5, 51, 66, 85, 98)$	112	$(3, 5, 51, 66, 86)$
113	$(3, 5, 51, 67, 87, 98)$	114	$(3, 5, 51, 67, 88)$
115	$(3, 5, 51, 68)$	116	$(3, 5, 52, 65, 87)$
117	$(3, 5, 52, 66)$	118	$(3, 11, 45, 45, 45, 50, 51, 65, 71, 98)$
119	$(3, 11, 45, 45, 45, 65, 71, 85, 98)$	120	$(3, 11, 45, 45, 45, 65, 71, 86)$
121	$(3, 11, 45, 45, 45, 67, 71, 98)$	122	$(3, 11, 45, 50, 51, 65, 87, 98)$
123	$(3, 11, 45, 50, 51, 65, 88)$	124	$(3, 11, 45, 50, 51, 66, 98)$
125	$(3, 11, 45, 65, 85, 87, 98)$	126	$(3, 11, 45, 65, 85, 88)$
127	$(3, 11, 45, 65, 86, 87)$	128	$(3, 11, 45, 66, 85, 98)$
129	$(3, 11, 45, 66, 86)$	130	$(3, 11, 45, 67, 87, 98)$
131	$(3, 11, 45, 67, 88)$	132	$(3, 11, 45, 68)$
133	$(5, 5, 5, 10, 18, 45, 50, 51, 51, 71, 98)$	134	$(5, 5, 5, 10, 18, 45, 51, 71, 85, 98)$
135	$(5, 5, 5, 10, 18, 45, 51, 71, 86)$	136	$(5, 5, 5, 10, 18, 45, 52, 71)$
137	$(5, 5, 5, 10, 19, 45, 51, 51, 71, 98)$	138	$(5, 5, 10, 11, 18, 45, 45, 50, 51, 71, 98)$
139	$(5, 5, 10, 11, 18, 45, 45, 71, 85, 98)$	140	$(5, 5, 10, 11, 18, 45, 45, 71, 86)$
141	$(5, 5, 10, 11, 18, 50, 51, 87, 98)$	142	$(5, 5, 10, 11, 18, 50, 51, 88)$
143	$(5, 5, 10, 11, 18, 85, 87, 98)$	144	$(5, 5, 10, 11, 18, 85, 88)$
145	$(5, 5, 10, 11, 18, 86, 87)$	146	$(5, 5, 10, 11, 19, 45, 45, 51, 71, 98)$
147	$(5, 5, 10, 11, 19, 51, 87, 98)$	148	$(5, 5, 10, 11, 19, 51, 88)$
149	$(5, 10, 11, 11, 18, 45, 45, 45, 50, 71, 98)$	150	$(5, 10, 11, 11, 18, 45, 50, 87, 98)$

j	$\mathbf{v}$	j	$\mathbf{v}$
151	$(5, 10, 11, 11, 18, 45, 50, 88)$	152	$(5, 10, 11, 11, 19, 45, 45, 45, 71, 98)$
153	$(5, 10, 11, 11, 19, 45, 87, 98)$	154	$(5, 10, 11, 11, 19, 45, 88)$
155	$(5, 10, 18, 39, 45, 45, 50, 51, 71, 98)$	156	$(5, 10, 18, 39, 45, 45, 71, 85, 98)$
157	$(5, 10, 18, 39, 45, 45, 71, 86)$	158	$(5, 10, 18, 39, 50, 51, 87, 98)$
159	$(5, 10, 18, 39, 50, 51, 88)$	160	$(5, 10, 18, 39, 85, 87, 98)$
161	$(5, 10, 18, 39, 85, 88)$	162	$(5, 10, 18, 39, 86, 87)$
163	$(5, 10, 19, 39, 45, 45, 51, 71, 98)$	164	$(5, 10, 19, 39, 51, 87, 98)$
165	$(5, 10, 19, 39, 51, 88)$	166	$(5, 10, 41, 45, 50, 51, 51, 71, 98)$
167	$(5, 10, 41, 45, 51, 71, 85, 98)$	168	$(5, 10, 41, 45, 51, 71, 86)$
169	$(5, 10, 41, 45, 52, 71)$	170	$(5, 10, 43, 45, 45, 51, 71, 98)$
171	$(5, 10, 43, 51, 87, 98)$	172	$(5, 10, 43, 51, 88)$
173	$(5, 10, 44, 50, 51, 51, 98)$	174	$(5, 10, 44, 51, 85, 98)$
175	$(5, 10, 44, 51, 86)$	176	$(5, 10, 44, 52)$
177	$(10, 11, 18, 39, 45, 45, 45, 50, 71, 98)$	178	$(10, 11, 18, 39, 45, 50, 87, 98)$
179	$(10, 11, 18, 39, 45, 50, 88)$	180	$(10, 11, 19, 39, 45, 45, 45, 71, 98)$
181	$(10, 11, 19, 39, 45, 87, 98)$	182	$(10, 11, 19, 39, 45, 88)$
183	$(10, 11, 41, 45, 45, 50, 51, 71, 98)$	184	$(10, 11, 41, 45, 45, 71, 85, 98)$
185	$(10, 11, 41, 45, 45, 71, 86)$	186	$(10, 11, 41, 50, 51, 87, 98)$
187	$(10, 11, 41, 50, 51, 88)$	188	$(10, 11, 41, 85, 87, 98)$
189	$(10, 11, 41, 85, 88)$	190	$(10, 11, 41, 86, 87)$
191	$(10, 11, 43, 45, 45, 45, 71, 98)$	192	$(10, 11, 43, 45, 87, 98)$
193	$(10, 11, 43, 45, 88)$	194	$(10, 11, 44, 45, 50, 51, 98)$
195	$(10, 11, 44, 45, 85, 98)$	196	$(10, 11, 44, 45, 86)$

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